Monopoly equilibrium and elasticity of substitution: a note on the existence of the equilibrium

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Abstract

We study the existence of a monopoly equilibrium in the bilateral mixed exchange framework introduced by Busetto et al.(2019). Non existence examples in which small traders have CES utility functions are provided and a link between the existence of an equilibrium and the degree of substitutability of the goods is explored. Therefore, the existence result is proved by introducing a sufficient assumption on the utilities of the small traders, stressing that we need them to be locally equivalent to linear utilities.
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1 Introduction

In their paper, Busetto et al. (2019) established a foundation for monopoly equilibrium in bilateral exchange. However, they left open the problems regarding the existence and optimality of such equilibrium. In the literature of strategic market games, initiated by Shapley and Shubik (1977), a lot of attention has been put on this topic. Busetto et al. (2011) initiated a line of research about existence in mixed models extending, in a way, Sahi and Yao (1989) existence result for finite economies in a Shapley windows model.
The problem with the existence in oligopoly models, specially models following the approach by Gabszewicz and Vial (1972), is that a discontinuity in the Walrasian price correspondence may arise, leading to non-existence of equilibria. In order to solve this problem, Busetto et al. (2011) used their assumption 4, that states that at least two large traders have interior endowments and the indifference curves passing through these points don't touch the axis (the assumption replicates the one in Sahi and Yao (1989)). Later on, they provided a refined version of the existence result in which they required a strongly connected set of commodities (Busetto et al., 2017), but assuming that small traders hold, in the aggregate, all the commodities present in the market. In the context of bilateral markets, Bloch and Ghosal (1997) provided an existence result in their model by assuming complementarity in the two goods for each agent.
However, the monopoly model presented in Busetto et al. (2019) fails to meet the assumptions stated in Busetto et al. (2011) and Busetto et al. (2017), which were needed

to prove an existence result. Therefore, it seems that additional assumptions are required in order to guarantee the existence of a monopoly equilibrium . Borrowing from the well known partial equilibrium studies on monopoly, we introduce a sufficient condition for the existence of a monopoly equilibrium based on the elasticity notion, closer to the approach of Bloch and Ghosal (1997) and Bloch and Ferrer (2001). In the latter, they consider a bilateral oligopoly in which every trader has a CES utility function, showing in their Lemma 1 that "the offers of traders on the two sides of the market are strategic complements(substitutes) if and only if the goods are substitutes (complements)" (p.85). We will initially consider that all small traders have an identical CES function, showing how the monopoly equilibrium behaves in the three limit cases for CES utility functions. We show that for a generic utility function form for the monopolist, the monopoly equilibrium fails to exist when small traders have Cobb-Douglas or Leontief utility function. In particular, the non-existence result for Cobb-Douglas utilities stresses how the assumption of small traders holding in the aggregate every good is crucial for some existence results, such as the one in Codognato and Julien (2013).

As Batra (1972) states, "we may conclude that a necessary condition for the monopoly equilibrium to exist is that both price elasticities of demand are greater than unity" (p.358). We need to even restrict this statement, claiming that the aggregate demand only needs to be locally similar to an aggregate demand derived from an homogeneous atomless sector which is endowed with a linear utility function to guarantee the existence of a monopoly equilibrium.

The outline of the existence proof follows the classical results in strategic market games. However, this is one of the first existence results in which a specific price selection is defined and for which an ϵ-equilibrium is proven to exist.

The model will follow from Busetto et al. (2019), i.e. a mixed version of a monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1, in which one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part.

The paper is organized as follows. In section 2, the mathematical model is introduced followed by a reminder of the notion of a monopoly equilibrium, in section 3. In sections 4 and 5, we compute the monopoly equilibrium when small traders have an identical CES utility function: we first consider the limit situations for CES utilities (i.e. Cobb-Douglas, Leontief and linear), followed by the general form, where we attempt to retrieve the limit situation results from the general case. In section 6 the existence theorem is proven, after introducing our sufficient condition. Finally, in section 7 we draw some conclusions and we suggest some further lines of research.

2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$ and $T \setminus T_0 = \{a\}$, i.e., the measure

space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.
A commodity bundle is a point in $\mathbb{R}^2_+$. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to \mathbb{R}^2_+$. We are considering a bilateral exchange economy, therefore with two commodities. We assume that the monopolist holds, without loss of generality good one, while small traders hold the second good, i.e.

Assumption 1. $\mathbf{w}_1(a) > 0$, $\mathbf{w}_2(a) = 0$ *and* $\mathbf{w}_1(t) = 0$, $\mathbf{w}_2(t) > 0$, *for each* $t \in T_0$.

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : \mathbb{R}^2_+ \to R$, satisfying the following assumptions.

Assumption 2. $u_t : \mathbb{R}^2_+ \to R$ *is continuous, strongly monotone, and strictly quasi-concave, for each* $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}^2_+$. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times \mathbb{R}^2_+ \to R$, *given by* $u(t, x) = u_t(x)$, *for each* $t \in T$ *and for each* $x \in \mathbb{R}^2_+$, *is* $\mathcal{T} \bigotimes \mathcal{B}$*-measurable.*

In order to state our last assumption, we need a preliminary definition. We say that commodities i, j stand in relation Q if there is a nonnull subset T^i of T_0, such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v^i_t(x_i) + v^j_t(x_j)$, for each $x \in \mathbb{R}^2_+$, and $\frac{dv^j_t(0)}{dx_j} = +\infty$, for each $t \in T^i$.[1] We can now introduce the last assumption.

Assumption 4. *Commodities 1 and 2 stand in relation* Q*.*

A price vector is a nonnull vector $p \in \mathbb{R}^2_+$. Moreover, we will denote by Δ the unit simplex, i.e. $\Delta = \{p\mathbb{R}^2_+ : p_1 + p_2 = 1\}$, and $\Delta \setminus \partial\Delta$ will denote the interior of Δ. Finally, we will write $P \in \mathbb{R}_+$ to intend the corresponding relative price for each $p \in \Delta \setminus \partial\Delta$, i.e. $P = \frac{p_1}{p_2}$, for some $(p_1, p_2) \in \Delta \setminus \partial\Delta$.
Let $\mathbf{X}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathcal{P}(\mathbb{R}^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$, $\mathbf{X}^0(t, p) = \text{argmax}\{u(x) : x \in \mathbb{R}^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in \mathbb{R}^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \{\int_{T_0} \mathbf{x}(t, p)\, d\mu : \mathbf{x}(\cdot, p)$ is integrable and $\mathbf{x}(t, p) \in \mathbf{X}^0(t, p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t, \cdot)$ is nonempty and single-valued, by Assumption 2, let $\mathbf{x}^0 : T_0 \times \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be the function such that $\mathbf{X}^0(t, p) = \{\mathbf{x}^0(t, p)\}$, for each $t \in T_0$ and for each $p \in \mathbb{R}^2_{++}$. A Walras equilibrium is a pair $(p, \mathbf{x})$, consisting of a price vector p and an allocation $\mathbf{x}$, such that $p\mathbf{x}(t) = p\mathbf{w}(t)$ and $u_t(\mathbf{x}(t)) \geq u_t(y)$, for all $y \in \{x \in \mathbb{R}^2_+ : px = p\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}$ for which there exists a price vector p such that the pair $(p, \mathbf{x})$ is a Walras equilibrium.

[1] In this definition, differentiability means continuous differentiability and is to be understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

3 Monopoly equilibrium

We now follow Codognato et al. (2019) in introducing their monopoly equilibrium concept.
Let $\mathbf{E}(a) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(a),\ i = 1,2\}$ denote the strategy set of atom a. We denote by $e \in \mathbf{E}(a)$ a strategy of atom a, where e_{ij}, $i, j = 1,2$, represents the amount of commodity i that atom a offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(a)$.

We then provide the following definitions.
Definition 1. *A square matrix A is said to be triangular if $a_{ij} = 0$ whenever $i > j$ or $a_{ij} = 0$ whenever $i < j$.*
Definition 2. *Given a strategy $e \in \mathbf{E}(a)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu + \sum_{i=1}^{2} e_{ij}\mu(a)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{2} e_{ji}\mu(a) \tag{1}$$

$, j = 1,2$.

The following proposition provides a necessary and sufficient condition for the existence of a market clearing price vector.
Proposition 1. *Under Assumptions 1, 2, 3, and 4, given a strategy $e \in \mathbf{E}(a)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

Proof. Let $e \in \mathbf{E}(a)$ be a strategy. Suppose that there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ and that the matrix E is not triangular. Then, it must be that $e_{12} = 0$. But then, we have that $\int_{T^2} \mathbf{x}^{01}(t,p)\, d\mu = 0$ as $\mu(T^2) > 0$, by (1). Consider a trader $\tau \in T^2$. We have that $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x_1} = +\infty$ as 2 and 1 stand in the relation Q and $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x_1} \leq \lambda \hat{p}^1$, by the necessary conditions of the Kuhn-Tucker theorem. Moreover, it must be that $\mathbf{x}^{02}(\tau,p) = \mathbf{w}^2(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $p\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x_2} = \lambda p_2$, by the necessary conditions of the Kuhn-Tucker theorem. But then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x_2} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, the matrix E must be triangular. Suppose now that E is triangular. Then, it must be that $e_{12} > 0$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1,2,\ldots$, which converges to a normalized price vector $\bar{p}$ such that $\bar{p}_1 = 0$. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t,p^n)\, d\mu\}$ diverges to $+\infty$, by Proposition 4. But then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{01}(t,p^n)\, d\mu > e_{12}\mu(a)$, for each $n \geq n_0$. Therefore, we have that $\int_{T_0} \mathbf{x}^{01}(t,p^{n_0})\, d\mu > e_{12}\mu(a)$. Let $q \in \Delta \setminus \partial\Delta$ be a price vector such that $\frac{q^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu}{q^1} = e_{12}\mu(a)$. Consider first the case where $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu = e_{12}\mu(a)$. Then, q is market clearing as it is market clearing for $j = 1$, by Proposition 3. Consider now the case where $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu \neq e_{12}\mu(a)$. Then, it must be that $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu < e_{12}\mu(a)$ as $\mathbf{x}^{01}(t,q) \leq \frac{q^2\mathbf{w}(t)}{q^1}$, for each $t \in T_0$. But then, we have that $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu < e_{12}\mu(a) < \int_{T_0} \mathbf{x}^{01}(t,p^{n_0})\, d\mu$. Let $O \subset \partial\Delta$ be a compact and convex set which contains p^{n_0} and q. Then, the correspondence $\int_{T_0} \mathbf{X}^0(t,\cdot)\, d\mu$ is upper hemicontinuous on O, by the argument used in the proof of Property (ii) in

Debreu (1982), p. 728. But then, the function $\{\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu\}$ is continuous on O as $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$, for each $p \in \Delta \setminus \partial\Delta$, by Proposition 1. Therefore, there is a price vector $p^* \in \Delta \setminus \partial\Delta$ such that $\int_{T_0} \mathbf{x}^{01}(t,p^*)\, d\mu = e_{12}\mu(a)$, by the intermediate value theorem. Then, p^* is market clearing as it is market clearing for $j = 1$, by Proposition 3. Hence, given a strategy $e \in \mathbf{E}(a)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular. ■

We denote by $\pi(e)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(a)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise. A price selection $p(e)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(a)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(a)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}\mathbf{x}^j(a,e,p) &= \mathbf{w}^j(a) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j}, \text{ if } p \in R^2_{++},\\ \mathbf{x}^j(a,e,p) &= \mathbf{w}^j(a), \text{ otherwise,}\end{aligned}$$

$j = 1,2,$

$$\begin{aligned}\mathbf{x}^j(t,p) &= \mathbf{x}^{0j}(t,p), \text{ if } p \in R^2_{++},\\ \mathbf{x}^j(t,p) &= \mathbf{w}^j(t), \text{ otherwise,}\end{aligned}$$

$j = 1,2$, for each $t \in T_0$.

Given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(a)$, traders' final holdings are expressed by the assignment $\mathbf{x}(a) = \mathbf{x}(a,e,p(e))$ and $\mathbf{x}(t) = \mathbf{x}(t,p(e))$, for each $t \in T_0$.

The following proposition shows that traders' final holdings are an allocation.

Proposition 2. *Under Assumptions 1, 2, 3, and 4, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(a)$, the assignment $\mathbf{x}(a) = \mathbf{x}(a,e,p(e))$ and $\mathbf{x}(t) = \mathbf{x}^0(t,p(e))$, for each $t \in T_0$, is an allocation.*

Proof. Let a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(a)$ be given. Suppose that E is not triangular. Then, we have that $\mathbf{x}(a) = \mathbf{x}(a,e,p(e)) = \mathbf{w}(a)$ and $\mathbf{x}(t) = \mathbf{x}(t,p(e)) = \mathbf{w}(t)$, for each $t \in T_0$ as $p(e) = 0$. Suppose that E is triangular. Then, we have that

$$\int_T \mathbf{x}^j(t)\, d\mu = (\mathbf{w}^j(a) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j})\mu(a) + \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu = \int_T \mathbf{w}^j(t)\, d\mu,$$

$j = 1,2$, as $p(e)$ is market clearing. Hence, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(a)$, the assignment $\mathbf{x}(a) = \mathbf{x}(a,e,p(e))$ and $\mathbf{x}(t) = \mathbf{x}^0(t,p(e))$, for each $t \in T_0$, is an allocation. ■

We can now provide the definition of a monopoly equilibrium.

Definition 3. *A strategy $\tilde{e} \in \mathbf{E}(a)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_a(\mathbf{x}(a,\tilde{e},p(\tilde{e})) \geq u_a(\mathbf{x}(a,e,p(e)),$$

for each $e \in \mathbf{E}(a)$.

3.1 Monopoly equilibrium under invertible demand

We now show how the monopoly equilibrium can be computed when the demand function $\int_{T_0} x^{01}(t,p)d\mu$ is invertible. This doesn't change the theoretical background of the definition just provided, it just aims to give support to the way in which the problem will be tackled in the next sections

In this situation, we want to show that finding the optimal bid for the monopolist is equivalent to obtaining the equilibrium bid as the demand computed at an optimal price.

We recall propositions 7 and 8 in Codognato et al. (2019).

Proposition 3. *Under Assumptions 1, 2, 3, and 4, the function* $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ *is invertible if and only, for each* $x \in R_{++}$, *there is a unique* $p \in \Delta \setminus \partial\Delta$ *such that* $x = \int_{T_0} \mathbf{x}^{0i}(t,p)\, d\mu$.

Proposition 4. *Under Assumptions 1, 2, 3, and 4, if the function* $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ *is invertible, then there exists a unique price selection* $\mathring{p}(\cdot)$.

Let $\mathring{p}(\cdot)$ denote the inverse of the function of $\int_{T_0} x^{01}(t,p)d\mu$. We prove the following proposition.

Proposition 5. *Under Assumptions 1, 2, 3, and 4, if the function* $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ *is invertible, then a strategy* $\tilde{e} \in \mathbf{E}(a)$ *such that* $\tilde{E}$ *is triangular is a monopoly equilibrium if and only if there exists a price* $\tilde{p} \in \Delta \setminus \partial\Delta$ *such that* $u_a(\mathbf{x}(a, e(\tilde{p}), \tilde{p}) \geq u_a(\mathbf{x}(a, e(p), p)$, *for each* $p \in \Delta \setminus \partial\Delta$. *Moreover,* $\tilde{p} = \mathring{p}(\tilde{e})$.

Proof. Suppose that $\tilde{e}$ is a monopoly equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. Clearly, $\tilde{p}$ is uniquely defined, by Propositions 3 and 4, as $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible. Suppose that there exists p' such that $u_a(\mathbf{x}(a, e(p'), p') \geq u_a(\mathbf{x}(a, e(\tilde{p}), \tilde{p})$. But then, letting e' be the unique strategy such that $p' = \mathring{p}(e')$, $u_a(\mathbf{x}(a, e', p(e')) \geq u_a(\mathbf{x}(a, \tilde{e}, \mathring{\tilde{e}})$, a contradiction sa $\tilde{e}$ is a monopoly equilibrium. But then, $u_a(\mathbf{x}(a, e(\bar{p}), (\bar{p})) \geq u_a(\mathbf{x}(a, e(p), p)$, for each $p \in \Delta \setminus \partial\Delta$.

Suppose now there exists a price $\tilde{p} \in \Delta \setminus \partial\Delta$ such that $u_a(\mathbf{x}(a, e(\tilde{p}), \tilde{p}) \geq u_a(\mathbf{x}(a, e(p), p)$, for each $p \in \Delta \setminus \partial\Delta$. Let $\tilde{e} = e(\tilde{p})$. Suppose that there exists $\bar{e}$ such that $u_a(\mathbf{x}(a, \bar{e}, (\mathring{p}(\bar{e})) \geq u_a(\mathbf{x}(a, \tilde{e}, p(\tilde{e})$. But then, letting $\bar{p} = \mathring{p}(\bar{e})$ we have $u_a(\mathbf{x}(a, e(\bar{p}), \bar{p}) \geq u_a(\mathbf{x}(a, e(\tilde{p}), \tilde{p})$, a contradiction. Therefore $\tilde{e} = e(\tilde{p})$ is a monopoly equilibrium.

■

The previous proposition tells that we can compute the monopoly equilibrium by first computing the optimal price and then finding the optimal bid that would result from the optimal price. Alternatively, proposition 5 states that if there is no optimal interior price, then there is no equilibrium.

4 Existence: Limit results

We will try now to give an existence result when the atomless part of the economy has an identical utility function, represented by a CES function in the form

$$u(x,t) = (ax_1^\rho + (1-a)x_2^\rho)^{\frac{1}{\rho}}.$$

The elasticity coefficient ρ plays a fundamental role in the analysis, so we will try and distinguish different situations depending on where the parameter lies.
First, we study what happens at the limit situations, i.e. when $\rho \to 0$, $\rho \to -\infty$ and $\rho = 1$.

4.1 Cobb-Douglas ($\rho \to 0$)

When the elasticity factor tends to 0, the utility function becomes a Cobb-Douglas, i.e. $u(x,t) = x_1^a x_2^{1-a}$.
In this situation, the demand function for good 1 becomes

$$x_1(P) = \frac{a}{P}$$

Then, the monopolist revenue in terms of good 2 is $P(e)e = a$. Therefore, a monopolist equilibrium doesn't exists as the induced utility function, i.e.

$$u(e,a,P(e)) = \begin{cases} (\mathbf{w}^1(a) - e, P(e)e) \ if \ e \in (0, w_1(a)] \\ \mathbf{w}(a) \ if \ e = 0 \end{cases} \tag{2}$$

is not continuous at $e = 0$, as $\lim_{e \to 0} P(e)e \neq 0$.

4.2 Linear utility($\rho = 1$)

If $\rho = 1$ (or tends to 1), then we approach the linear utility case, i.e. $u(x,t) = ax_1 + (1-a)x_2$.
In this case, given the corner endowments, the first order conditions of the utility maximization problem directly give the value for the relative price, i.e. $P = \frac{a}{1-a}$. Therefore, the monopolist becomes price taker as well, and the monopolist equilibrium coincides with competitive equilibrium. As a consequence, the monopoly equilibrium exists.

4.3 Leontief ($\rho \to -\infty$)

The final limit case is the one in which $\rho \to -\infty$. In this situation, the utility becomes $u(x,t) = min\{\frac{x_1}{a}, \frac{x_2}{1-a}\}$. The demand for good one becomes then

$$P(x_1) = \frac{a + ax_1 - x_1}{ax_1}$$

In the same way as the Cobb Douglas, the induced utility function for the monopolist is not continuous, as $\lim_{e \to 0} P(e)e = 1 \neq 0$. Therefore, the monopoly equilibrium doesn't exist.

4.4 The general case for CES utilities

Let's now move to the general case. Every small trader solves the maximization problem

$$\max u(x,t) = (ax_1^{\rho} + (1-a)x_2^{\rho})^{\frac{1}{\rho}} \; s.t. Px_1 + x_2 = 1$$

which, for non degenerate cases, leads to the following demand function:

$$x_1(P,t) = \frac{1}{P + (\frac{1-a}{a}P)^{\frac{1}{1-\rho}}} \tag{3}$$

First, we can check under which values of ρ this demand function satisfies relation Q, i.e. $\lim_{x_1 \to 0} \frac{\partial u(x,t)}{\partial x_1} = +\infty$.

$$\frac{\partial u(x,t)}{\partial x_1} = \frac{1}{\rho} a x_1^{\rho-1} (ax_1^{\rho} + (1-a)x_2^{\rho})^{\frac{1}{\rho}}$$

We can clearly see that the limit of the partial utility goes to infinity as x_1 goes to 0 when $\rho < 1$.
Now, since the demand function can't be generically inverted to obtain a demand function in the form $P(x_1)$, we will consider the monopolist problem from a price setting perspective. We can observe that when $\lim_{P \to +\infty} x_1(P,t) = 0$.

Remind that the CES utility function have the property that the elasticity is constant, i.e. $-\frac{\partial \frac{dx_1}{dx_2}}{dp} \frac{P}{\frac{x_1}{x_2}} = \frac{1}{1-\rho} = \phi$. So we can rewrite all of these relations in terms of ϕ.

The monopolist observe the small traders demand function and solves the following problem:

$$\begin{aligned} \max_P \; & u(x,a) \\ s.t. \; & Px_1 + x_2 = P \\ & 1 - x_1(a) = x_1(P,t) \end{aligned}$$

First, we can rewrite the first constraint as $x_2 = (1 - x_1(a))P = x_1(P,t)P$.
It may be worth noticing that prices are bounded below. This is implicitly stated in the second constraint, as prices must be such that $x_1(P,t) \leq 1$, and therefore

$$\frac{1}{P + (\frac{1-a}{a}P)^{\phi}} \leq 1.$$

Now, we can plug the constraints into the utility function, and the problem reduces to

$$\max_P \; u(x_1(a,P), x_2(a,P)) = u(1 - x_1(P,t), Px_1(P,t)) \tag{4}$$

4.5 Inelastic demand and non existence of monopoly equilibrium

Before going into the analysis of the first order conditions for this problem, it is worth noticing that $Px_1(P,t)$ may not go to 0 when the relative price diverge, i.e. when the bid of the monopolist goes to 0. In particular, when $0 < \phi < 1$, $\lim_{P\to 0} x_1(P,t) = 1$. This created the discontinuity we encountered in the previous examples. We can therefore state the following proposition.

Proposition 6. *If $u(x,t)$ is a CES with elasticity parameter with $0 < \phi < 1$, for each $T \in T_0$, then there is no monopoly equilibrium.*

Proof. Monopolist final allocation will be in the form $x_a(e, P(e)) = (1 - e(P), Pe(P))$. Moreover, $Pe(P) = Px_1(P,t) = \frac{P}{P+(\frac{1-a}{a}P)^{\phi}}$. This expression goes to 1 when $P \to +\infty$, as we are assuming $0 < \phi < 1$. But then, $\lim_{P\to+\infty} x_a(e, P(e)) = (1,1)$, as $e(P) = x_1(P,t)$ and $x_1(P,t) \to 0$ when $P \to +\infty$. Moreover, $(1,1) \succ_a x_a(e, P(e))$, for each $P \in \mathbb{R}_+$ (i.e. $p \in \Delta \setminus \partial\Delta$). However, at the limit, i.e. when $e = 0$, $x_a(0) = w_a = (1,0)$. Therefore, there is no optimal strategy for the monopolist, in the sense that the it is always optimal to increase the price (reduce the bid). Hence, there is no monopoly equilibrium ■

4.6 Elastic demand

We can now focus on the general solution of the maximization problem stated in 4. The first order condition is

$$\frac{\partial u}{\partial P} = -\frac{\partial u(x,a)}{\partial x_1(P,a)}\frac{\partial x_1(P,t)}{\partial P} + \frac{\partial u(x,a)}{\partial x_2(P,a)}(x_1(P,t) + P\frac{dx_1}{dP})$$

Expanding the constant elasticity relation, we can write

$$\frac{dx_1}{dP} = -\frac{\phi x_1(1-Px_1)}{P} - x_1^2$$

Therefore, rearranging the terms, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial P} &= -\frac{\partial u(x,a)}{\partial x_1(P,a)}(\frac{dx_1(P,t)}{dP}) + \frac{\partial u(x,a)}{\partial x_2(P,a)}\frac{d[Px_1(P,t)]}{dP} \\
&= -\frac{\partial u(x,a)}{\partial x_1(P,a)}(-\frac{\phi x_1(1-Px_1)}{P} - x_1^2) + \frac{\partial u(x,a)}{\partial x_2(P,a)}[x_1(P,t)^2(1-\phi)(\frac{1-a}{a}P)^{\phi}] \\
&= x_1(P,t)^2[\frac{\partial u(x,a)}{\partial x_1(P,a)}(\frac{\phi}{P}(\frac{1-a}{a}P)^{\phi} + 1) + \frac{\partial u(x,a)}{\partial x_2(P,a)}(1-\phi)(\frac{1-a}{a}P)^{\phi}] \qquad (5)
\end{aligned}$$

Analyzing this expression, we can already find an interesting result, that is that the marginal change utility for the monopolist for a price change is decreasing in the elasticity parameter. We prove this result in the following proposition.

Proposition 7. *If $\phi_1 \geq \phi_2 > 1$, then $\frac{\partial u}{\partial P}(\phi_1) \leq \frac{\partial u}{\partial P}(\phi_2)$*[2]*.*

[2] With a little abuse of notation, we denote by $\frac{\partial u}{\partial P}(\phi_1)$ the derivative of the derivative of the induced utility function for the monopolist when she faces an homogeneous atomless sector in which all traders have a CES utility function with elasticity parameter ϕ_1

Proof. Suppose $\phi_1 \geq \phi_2 > 1$. Then, it is immediate to see that $x_1(\phi_1, P, t) \leq x_1(\phi_2, P, t)$, for each P, from the expression of the demand function (see 3). But then, $x_1(\phi_1, P, a) \geq x_1(\phi_2, P, a)$ and $x_2(\phi_1, P, a) \leq x_1(\phi_2, P, a)$, for each P, as $x(a, P) = (1 - x(P, t); Px_1(P, t))$. Therefore, $\frac{\partial u(x,a)}{\partial x_1(P,a)}|_{x_1(P,a)=x_1(\phi_1,P,a)} \leq \frac{\partial u(x,a)}{\partial x_1(P,a)}|_{x_1(P,a)=x_1(\phi_2,P,a)}$ and $\frac{\partial u(x,a)}{\partial x_2(P,a)}|_{x_2(P,a)=x_2(\phi_1,P,a)} \geq \frac{\partial u(x,a)}{\partial x_2(P,a)}|_{x_2(P,a)=x_2(\phi_2,P,a)}$, as u is strictly concave, by Assumption 2. Hence, by incorporating the previous disequations for equation 5, if $\phi_1 \geq \phi_2 > 1$, then $\frac{\partial u}{\partial P}(\phi_1) \leq \frac{\partial u}{\partial P}(\phi_2)$. ■

To derive the result for linear utilities, we will show in the following proposition that when the elasticity parameter ϕ goes to infinity, then the optimal monopoly price will equate the walrasian/paretian price.

Proposition 8. *Consider a pure exchange economy such that each trader $t \in T_0$ has a CES utility function with parameter ϕ, then when $\phi \to +\infty$ the monopoly equilibrium will coincide with the walrasian equilibrium.*

Proof. We consider again the first order condition, expressed in 5, and we will put it to be greater or equal to 0, i.e.

$$x_1(P,t)^2[\frac{\partial u(x,a)}{\partial x_1(P,a)}(\frac{\phi}{P}(\frac{1-a}{a}P)^{\phi} + 1) + \frac{\partial u(x,a)}{\partial x_2(P,a)}(1-\phi)(\frac{1-a}{a}P)^{\phi}] \geq 0$$

Rearranging the terms, we get that

$$-\frac{\frac{\partial u(x,a)}{\partial x_1(P,a)}}{\frac{\partial u(x,a)}{\partial x_2(P,a)}} \leq P\frac{(1-\phi)(\frac{1-a}{a}P)^{\phi}}{\phi(\frac{1-a}{a}P)^{\phi} + P} \tag{6}$$

In particular, we may notice that the right hand side of the previous disequation goes to P as $\phi \to +\infty$. If we rewrite the previous expression as an equation, then we obtain the well known relation for a walrasian economy, i.e. $\frac{\frac{\partial u(x,a)}{\partial x_1(P,a)}}{\frac{\partial u(x,a)}{\partial x_2(P,a)}} = P$; therefore proving the fact that when the elasticity goes to infinity, i.e. when the CES utilities tend to linear form utility, the monopoly equilibrium will converge to the walrasian equilibrium. ■

4.7 Example with heterogeneous atomless sector

We end this section with an instructive example that extends Proposition 6. We show that even when only a subset of the small traders has an inelastic CES utility function, we end up with a negative result.

Example 1. *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, 3 and 4. $T_0 = [0,1]$, $T_1 = 2$, T_0 is taken with Lebesgue measure and $\mu(2) = 1$, $\mathbf{w}_t = (0,1)$ for each $t \in T_0$, $u_t(x) = x_1 x_2$ for each $t \in [0, \frac{1}{10}]$, $u_t(x_1, x_2) = \sqrt{x_1} + x_2$ for each $t \in [\frac{1}{10}, 1]$, $\mathbf{w}_2 = (1,0)$, $u_2 = 20x_1 + \frac{1}{10} ln x_2$. Then, there is no monopoly equilibrium.*

Proof. The demand function for good 1 for each $t \in [0, \frac{1}{10})$ is given by $x_1(t, P) = \frac{1}{2P}$, while the the demand function for good 1 for each $t \in [\frac{1}{10}, 1]$ is given by $x_1(t, P) = \frac{1}{4P^2}$. Therefore, the aggregate demand function for good 1 is given by

$$\int_{T_0} x_1(t,p)d\mu = \int_0^{\frac{1}{10}} x_1(t,P)d\mu + \int_{\frac{1}{10}}^1 x_1(t,P)d\mu = \frac{1}{10}\frac{1}{2P} + \frac{9}{10}\frac{1}{4P^2} = \frac{2P+9}{40P^2}.$$

Then, $p(e) = p(\int_{T_0} x_1(t,p)d\mu) = \frac{1+\sqrt{1+360e}}{40e}$. But then, the induced utility for the monopolist $u_2(x_1(e,P), x_2(e,P) = (1-e, p(e)e) = 20(1-e) + \frac{1}{10}ln\frac{1+\sqrt{1+360e}}{40}$. Therefore, the first order condition for the maximization of the utility of the monopolist are

$$\frac{du_2(e))}{de} = -20 + \frac{18}{360e + 1 + \sqrt{1+360e}}.$$

This expression is clearly negative for each value of $e \in (0,1]$. Moreover, $\lim_{e\to 0}(x_1(e), x_2(e)) \neq w_2$ as $\lim_{e\to 0} x_2(e) = \lim_{e\to 0} P(e)e = \frac{1}{20}$. Hence, there is no monopoly equilibrium. ■

4.8 Discussion

In this paper, we tried to link results about existence and behaviour of a monopoly equilibrium with the elasticity of substitution between the two goods.
In order to do that, we consider a pure exchange economy in which all traders in the atomless part has an identical CES function, that guarantees that the aggregate demand function will preserve the constant elasticity property.
The first proposition is a counterpart in our framework of the result in monopoly theory, stating that "[...] the profit maximizing monopolist produces an output where the marginal revenue equals positive marginal cost, and the former is positive only if the elasticity of demand exceeds unity" (Batra 1971, pag.358). In our context, we state that the elasticity of substitution must exceed unity in order for the monopoly equilibrium to exist. In particular, the non existence is derived from the fact that when the elasticity of substitution is sufficiently low, the monopolist can exploit indefinitely the small traders, but at the limit he's left with her own endowment, which is strictly worse that what she could have got if she kept on decreasing her offer (increasing the price).

In the two limit situations, i.e. Cobb-Douglas and Leontief utility functions, the interpretation appears even "cleaner". For Leontief utility function, the two goods are perfect complements, therefore the monopolist can attain his maximum market power as small traders will always be incentivized to send to the market almost their whole stock of good in order to exchange it for a small quantity from the good the monopolist owns.
In the Cobb-Douglas case, we can see that the small traders demand for the good they own is completely inelastic. This means that the monopolist will always get in return a fixed amount of the good the small traders own, no matter what his bid is. Therefore, the monopolist here is always incentivized to reduce his bid.

To conclude our analysis of the first proposition, we will add two details to it. First, the result states the non existence of a monopoly equilibrium in the sense that there

doesn't exist an optimal strategy for the monopolist, not that the only possible equilibrium is autarchy. This is mostly due to the fact that the low elasticity of substitution makes the induced payoff of the monopoly discontinuous, and therefore we can't have a solution to the maximization problem. The important feature that drives the discontinuity is the model setup, in particular the fact that we have identical corner endowment for each small traders, where none of them holds any amount of the good owned by the monopolist.
The second important fact is that this proposition doesn't require any additional assumption for the behaviour of monopolist utility, as it holds for a generic utility function for the monopolist.

The second proposition can also be considered as a generalization for the standard result in partial equilibrium, that is the well known mark-up formula

$$MR(1 + \frac{1}{\phi}) = MC.$$

Clearly, if the elasticity of the demand goes to infinity, than we get the standard competitive result. Our proposition states the same result in a context of bilateral exchange. Here the interpretation is that the closer the goods to the situation of perfect substitutes, formally linear utility form, the lesser market power the monopolist has. In the limit, when goods are perfect substitutes, the relative price of the goods is fixed by the small traders via their demand (as it is infinitely elastic) and therefore the monopolist has no market power in manipulation the price, which in turn brings the equilibrium to be equal to the walrasian one.
Finally, it is worth noticing that in this situation the existence of a non autarchic equilibrium is established as we know from standard results in general equilibrium theory that guarantees the existence of a walrasian equilibrium in this framework.

5 Existence of a monopoly equilibrium

The previous section showed that the monopoly equilibrium may fail to exist in the context of an inelastic demand. Therefore, we'll provide a proof for the existence of a monopoly equilibrium which takes into account this feature.
Before introducing our new assumption, we need to introduce the following definition.
Definition 4. *Let u_1 and u_2 be two utility function satisfying Assumption 2. We say that the two utilities are locally equivalent at $\bar{x} \in \mathbb{R}^2_+$ if there exists a sequence of prices $P^n \in \mathbb{R}_+$ for which the corresponding demands $x_1(P^n)$ coincide and both converge to $\bar{x}$.*

Finally, in order to take into account the elasticity constraint, we introduce the following assumption.
Assumption 5. *There exist $\alpha \in (0,1)$ such that u_t is locally equivalent to a linear utility function with parameter α at $\mathbf{w}_t = (0,1)$, for each $t \in T_0$,.*

This assumption requires the utility functions of each small traders to be locally equivalent to a linear utility function.

However, the first part of the proof will be given from a general perspective, as it holds for a more general framework[3].

Theorem 1. *Under Assumptions 1, 2, 3, 5 there exists a monopoly equilibrium.*

Proof. From now on, since the demand functions are homogeneous of degree 0, instead of considering non negative price vectors, we will consider price vectors $p_\epsilon \in \Delta = \{p \in \mathbb{R}^2_+ : p^1 + p^2 = 1\}$. We will denote the set of strictly positive prices as $\Delta \setminus \partial\Delta$.
We show now a proposition about the aggregate demand function $\int_T \mathbf{x}^{01}(t, p_\epsilon) d\mu(t) : \Delta \setminus \partial\Delta \to \mathbb{R}_+$.

Lemma 1. *The aggregate demand function $\int_T \mathbf{x}^{01}(t, p_\epsilon) d\mu(t) : \Delta \setminus \partial\Delta \to \mathbb{R}_+$ is an onto continuous function.*

Proof. The correspondence $\int_{T_0} \mathbf{X}^0(t, \cdot)\, d\mu$ is upper hemicontinuous, by the argument used in the proof of Property (ii) in Debreu (1982), p. 728. But then, the function $\{\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu\}$ is continuous as $\int_{T_0} \mathbf{X}^0(t, p_\epsilon)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p_\epsilon)\, d\mu$, for each $p_\epsilon \in \Delta \setminus \partial\Delta$, by Proposition 1 in the other paper.
To prove that $\int_T \mathbf{x}^{01}(t, p_\epsilon) d\mu(t) : \Delta \setminus \partial\Delta \to \mathbb{R}_+$ is onto, we need to show that for each $e \in \mathbf{E}(a)$ there exists a market clearing price $p_\epsilon \in \Delta \setminus \partial\Delta$. First, let $\epsilon > 0$. Then, let $e \geq 0$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$ such that $\bar{p}^1 = 0$. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu\}$ diverges to $+\infty$, by Proposition 4 in the other paper. But then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu > e + \epsilon$, for each $n \geq n_0$. Therefore, we have that $\int_{T_0} \mathbf{x}^{01}(t, p^{n_0})\, d\mu > e + \epsilon$. Let $q \in \Delta \setminus \partial\Delta$ be a price vector such that $\frac{q^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu}{q^1} = e + \epsilon$. Consider first the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu = e + \epsilon$. Then, q is market clearing as it is market clearing as it satisfies 1. Consider now the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu \neq e + \epsilon$. Then, it must be that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e + \epsilon$ as $\mathbf{x}^{01}(t, q) \leq \frac{q^2 \mathbf{w}(t)}{q^1}$, for each $t \in T_0$. But then, we have that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e + \epsilon < \int_{T_0} \mathbf{x}^{01}(t, p^{n_0}_\epsilon)\, d\mu$. Let $O \subset \Delta \setminus \partial\Delta$ be a compact and convex set which contains $p^{n_0}_\epsilon$ and q. . Therefore, there is a price vector $p^*_\epsilon \in \Delta \setminus \partial\Delta$ such that $\int_{T_0} \mathbf{x}^{01}(t, p^*)\, d\mu = e + \epsilon$, by the intermediate value theorem. . Hence, given a strategy $e \in \mathbf{E}(a)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$. ■

We can now start giving the existence result in the perturbed game we just defined.
Given $\epsilon > 0$, define a map from market clearing price vectors into monopolist actions, which is a restriction of the aggregate demand function, namely $\mathring{e} : A_\epsilon \subseteq \Delta \setminus \partial\Delta \to (0, \mathbf{w}^1(\mathbf{a})]$, with $A_\epsilon = \{p_\epsilon \in \Delta \setminus \partial\Delta : \epsilon \leq \int_{T_0} \mathbf{x}^{01}(t, p_\epsilon) d\mu(t) \leq \mathbf{w}^1(a) + \epsilon\}$.

Lemma 2. *The mapping $\mathring{e}(p_\epsilon) : A_\epsilon \subseteq \Delta \setminus \partial\Delta \to (0, \mathbf{w}^1(\mathbf{a})]$ is a continuous function and a closed mapping.*

[3]For example, the first part of the proof would hold even replacing Assumption 5 back with Assumption 4

Proof. $\mathring{e}(p_\epsilon)$ is a function as it is a restriction of the aggregate demand function $\int_{T_0} \mathbf{x}^{01}(t, p_\epsilon)d\mu(t)$: $\Delta \setminus \partial\Delta \rightarrow \mathbb{R}_+$. The function is also continuous as restrictions preserves continuity. Moreover, the set A_ϵ is closed as it is a preimage of a closed set via a continuous function. Moreover, A_ϵ is bounded as $A_\epsilon \subset \Delta$, which is a compact set. Therefore, A_ϵ is compact, as it is closed and bounded. Hence, $\mathring{e}_\epsilon(p)$ is a closed map, by Theorem 4.95 in Lee, John M. (2011) ■

We can give an initial characterization of the (restricted) inverse correspondence $\pi_\epsilon(e)$: $(0, \mathbf{w}^1(a)] \twoheadrightarrow B \subseteq A_\epsilon$.

Lemma 3. *The correspondence $\pi_\epsilon(e)$ is non-empty, compact valued and upper hemicontinuous.*

Proof. $\mathring{e}_\epsilon(p)$ has a closed graph, by Lemma 2. Then, $\pi_\epsilon(e)$ is an upper hemicontinuous correspondence, by Theorem 17.7 in Aliprantis and Border (2006). Moreover, $\pi_\epsilon(e)$ is non-empty, by Lemma 1. Following the previous arguments, $\pi_\epsilon(e)$ is bounded as $\pi_\epsilon(e) \in A_\epsilon \subseteq \Delta$, and it is closed as it is the preimage of $\{e + \epsilon\}$ via the aggregate demand function (which is continuous). Hence, $\pi_\epsilon(e)$ is a compact valued, upper hemicontinuous correspondence. ■

Let now $\tilde{p}_\epsilon(e) = \text{argmax}_{p_\epsilon \in \pi_\epsilon(e)} p^1_\epsilon(e)$. This is a well defined selection as $\pi_\epsilon(e)$ is compact valued, by Lemma 3.

We now give a lemma that states an additional property for this selection.

Lemma 4. *The price selection $p(e) = max_p \pi(e)$, expressed in terms of relative prices, i.e. $P(e)$, is decreasing for each $e > 0$.*

Proof. Let $e' > e''$ and suppose $P(e') \geq P(e'')$. Consider a restriction of the aggregate demand function $\int_{T_0} x^{01}(t, p)d\mu(t)$ by restrincting the domain of this function to the set $[P(e'), +\infty)$. Moreover, we know that the aggregate function (and therefore the restriction) is continuous and $\lim_{P \rightarrow +\infty} \int_{T_0} x^{01}(t, p)d\mu(t) = 0$. But then, there exists a $P' \geq P(e'')$ such that $\int_{T_0} x^{01}(t, p')d\mu(t) = e''$, by the Intermediate Value Theorem. But then, there exists $p' \in \pi(e'')$ with $P' = \frac{p'^1}{p'^2}$ and $P' > P(e'')$, a contraddiction. Therefore, $P(e'') > P(e')$.
Hence, the price selection $p(e) = max_p \pi(e)$ is decreasing in e. ■

We can now define, in a similar way, $u_a(\mathbf{x}(a, e, \tilde{p}_\epsilon(e)) = \max_{p \in \pi_\epsilon(e)} u_a(\mathbf{x}(a, e, p_\epsilon(e))$. We can now provide a characterization for this induced payoff function.

Lemma 5. *The induced payoff function, given by $u_a(\mathbf{x}(a, e, \tilde{p}_\epsilon(e))$ is upper semicontinuous.*

Proof. The correspondence $\pi_\epsilon(e)$ is compact valued and upper hemicontinuous, by Lemma 3. The utility function $u_a(\mathbf{x}(a, e, p))$ is continuous by Assumption 2. Hence, $u_a(\mathbf{x}(a, e, \tilde{p}_\epsilon(e))$ is upper semicontinuous by Lemma 17.30 in Aliprantis and Border (2006). ■

Finally, we apply Luenberger's version of Weierstrass theorem to finally obtain the existence result for the perturbed version of the economy.

Lemma 6. *An ϵ-monopoly equilibrium exists.*

Proof. From the definition of monopoly equilibrium, we can see that a monopoly equilibrium action $\tilde{e}_\epsilon$ is such that $\tilde{e}_\epsilon \in argmax_{e \in \mathbf{E}} u(x(e, p_\epsilon(e), a)$, with respect to a price selection $p_\epsilon(\cdot)$. Let this price selection be $\tilde{p}_\epsilon(e) = \text{argmax}_{p_\epsilon \in \pi_\epsilon(e)} p^1_\epsilon(e)$. But then, $u_a(\mathbf{x}(a, e, \tilde{p}_\epsilon(e))$ is upper semicontinuous, by Lemma 5. Therefore, this function achieves a maximum in its domain, i.e. $[0, \mathbf{w}^1(a)]$, by Theorem 1 in (Luenberger, 1970, p. 40). Hence, an ϵ-monopoly equilibrium exists. ∎

Consider a sequence $\{\epsilon^n\}$ with $\lim_{n\to\infty} \epsilon^n = 0$. For each $\{\epsilon^n\}$, there exists an optimal $\tilde{P}^n_\epsilon$, by Lemma 6.
$\tilde{P}^n_\epsilon$ is bounded above for each ϵ, as the set of feasible prices is bounded above by $P^n_\epsilon(0) = \{P \in R_+ : \int_{T_0} x_1(P, t) d\mu = \epsilon\}$. However, $\lim_{n\to\infty} P^n_\epsilon(0) = +\infty$, as $\lim_{n\to\infty} \epsilon^n = 0$. Moreover, when $\{\epsilon^n\} \to 0$, the final allocation for the small traders is $x_1(P^n_\epsilon, t) = (x(P^n_\epsilon, t), x_2(P^n_\epsilon, t))$, which converges to $x(P^n_\epsilon, t) = (0, 1)$ when $P^n_\epsilon \to (1, 0)$. But then, there exists a subsequence $\{P^{n_k}_\epsilon\}$ for which the demand $x_1(P^{n_k}_\epsilon, t)$ will coincide with the demand function of a linear utility function, by Assumption 5. Therefore, there exists $\bar{n}_k$, such that $P^{\bar{n}_k}_\epsilon < \frac{\alpha}{1-\alpha} \leq P^{n_{\bar{k}+1}}_\epsilon$, as the sequence diverges and $\alpha \in (0, 1)$. But then, $\int_{T_0} x_1(P^{n_k}_\epsilon) d\mu = 0$ for each $n_k \geq \bar{n}_k$. Then, $x_a(e, P^{n_k}_\epsilon(e_{n_k})) = (\mathbf{w_1}(\mathbf{a}) - e_{n_k}, P^{n_k}_\epsilon(e_{n_k}) e_{n_k}) = (\mathbf{w_1}(\mathbf{a}) - e_{n_k}, 0)$, for each $n_k \geq \bar{n}_k$. Therefore, the optimal bid will always be either 0 or greater then $e_{\bar{n}_k}$. Assuming that the endowment for the monopolist is not pareto efficient, we must have that that the optimal bid for the monopolist is bounded below by $e_{\bar{n}_k}$. But then, there exists a subsubsequence $\tilde{e}_{n_{k_m}}$ that converges to a point $\tilde{e} \in [e_{\bar{n}_k}, \mathbf{w_1}(\mathbf{a})$. This completes the proof.
Hence, a monopoly equilibrium exists.

∎

6 Conclusion

In this paper, we studied the problem of the existence of a monopoly equilibrium, linking it with the elasticity of substitution of small traders utilities.
We gave an existence result for the framework introduced in Busetto et al. (2019), introducing a sufficient local condition for the utilities in the small trader sector that guarantees the existence of such an equilibrium. We showed that there is a link between the elasticity of substitution of the aggregate demand and the existence of an equilibrium by considering a situation in which the small traders have a generic CES utility function. Therefore, we introduced a sufficient condition that guarantees the existence of equilibrium, taking into account this feature.
We extend the well known result in partial equilibrium that monopolist will produce in the inelastic portion of the demand curve, by requiring that the small traders have preference which are locally equivalent to linear utilities. This last assumption is very

specific and quite demanding, but it also arises naturally after we considered the previous examples.
A more general consideration of the existence problem of a monopoly equilibrium and its relation to substitutability/complementarity notions, expanding on the arguments by Bloch and Ferrer (2001) and Bloch and Ghosal (1997), is left for further work. In particular, relaxing Assumption 5 by allowing utilities to be locally equivalent to a generic CES would seem the most direct extension to our proof.

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