Disputes, Debt and Equity *

Alfred Duncan† Charles Nolan‡
University of Kent University of Glasgow

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We show how the prospect of disputes over firms' revenue reports promotes debt financing over equity. These findings are presented within a costly state verification model with a risk averse entrepreneur. The prospect of disputes encourages incentive contracts that limit penalties and avoid stochastic monitoring, even when the lender can commit to stochastic monitoring strategies. Consequently, optimal contracts shift away from equity and toward standard debt. For a useful special case of the model, closed form solutions are presented for leverage and consumption allocations under efficient debt contracts. Some empirical implications of the theory are pursued.

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†Corresponding author. Email: `a.j.m.duncan@kent.ac.uk`. School of Economics, University of Kent, Canterbury CT2 7NP, United Kingdom.

‡Email: `Charles.Nolan@glasgow.ac.uk`

INTRODUCTION AND OVERVIEW

What form should an optimal external finance contract take to handle the prospect of audit errors? Our answer is that standard debt contracts are often optimal. Thus, we propose a theory of debt and limited liability based on inaccurate auditing.

We conduct the analysis in a costly state verification environment which typically implies that equity contracts are optimal. However, that conclusion is shown to rest crucially on the efficacy of the audit technology. We introduce wrongful penalties through an imperfect audit technology although our results carry over to other situations where the lender or bankruptcy court might erroneously dispute the borrowers revenue report.

In the model we present, a contract is an enforceable agreement between a risk-averse borrower and a risk-neutral lender covering the amount borrowed (leverage), an audit strategy dependent on the state of nature declared by the borrower, and pay-offs to the parties to the contract given the declared state of nature and the findings of any audit. Perceived misreporting may be penalized and truth-telling rewarded. Whether or not the audit is *perfect*–capable without exception of identifying the true underlying state–can make all the difference to the form of the optimal contract.

With perfect audits, the optimal contract will usually employ stochastic audits with large penalties. That way, the parties to the contract conserve on the cost of audits whilst ensuring truth-telling; penalizing those who misreport, and rewarding those who do not. The level of borrowing complements these decisions; higher leverage, other things constant, boosts the borrowers expected pay-off but is accompanied by larger penalties for misreporting. The upshot is that the contract delivers substantial risk-sharing: That is, the borrower's consumption is relatively insensitive, and the lender's return is relatively sensitive, to reported income. Such a contract has key properties of equity finance.

Imperfect audits complicate things. Stochastic audits and large penalties run the risk of penalizing truth-telling agents when project returns are indeed low, whilst higher leverage, ceteris paribus, increases both the expectation and variance of borrowers consumption. Risk-averse borrowers at the margin fret more about the cost of having a truthful low report overturned than they welcome the prospect of an overturned high report. The optimal contract, therefore, has lower equilibrium borrowing relative to the perfect audits case and lower penalties.[1] Moreover, audits are employed only when the desire for risk-sharing is especially intense, i.e., in sufficiently bad states of the world. In other states, the borrower will absorb marginal income risk, avoiding costly, potentially erroneous audits. Such a contract, auditing and risk sharing with the lender only in low states, is akin to a standard debt contract.

[1]Our model is static, and our penalties are just units of the consumption good, but this reasoning holds under alternative settings where alternative enforcement schemes such as non-pecuniary penalties or exclusion may be applied. When disputes are possible, even honest borrowers prefer the penalties for dishonesty to be smaller, all else equal.

IMPERFECT AUDITS, DETERMINISTIC INCENTIVE REGIMES AND LEVERAGE

When audits are perfect but costly, the marginal benefit of increasing the probability of auditing any individual state is high when the probability of audit is low. However, there are limits on how much auditing is desirable under perfect auditing. Theorem 1 shows that the application of perfect audits with certainty is wasteful; resources are saved by constructing a noisy audit signal using a lottery. The resulting imperfect audit technology is less expensive but still ensures truthful reporting. Optimal contracts under perfect auditing employ stochastic audits to reward truth-telling and, combined with high penalties, deter fraudulent reporting. Hence, the entrepreneur's consumption is protected from random fluctuations in firm value, in a way that resembles equity finance. Theorem 1 thus explains why deterministic audit schemes promote the use of 'imperfect' audits.

In the model with costly, imperfect audit technology, entrepreneurs may experience one of a number of outturns for their project. Stochastic auditing and a penalty may deter some, but not all, from misreporting. In other words, the deterrence of marginal fraudulent reports does not imply the deterrence of major fraudulent reports; the single-crossing property is absent. Certain audits are more than sufficient to deter low-income agents from misreporting, but any decrease in audit probability might encourage high-income agents fraudulently to default.

Thus, with penalties limited, higher audit probability may be part of the response. However, even as the probability of audit approaches one, the marginal benefit of increasing the probability of audit remains strictly positive. Such increases facilitate a given insurance plan with smaller penalties, reducing the costs of wrongful errors. So, in low states, where the marginal utility of the entrepreneur is high, the insurance benefits from auditing outweigh the cost. The optimal audit probability following low reports is one, resembling a standard debt contract. That is the key result proved in Theorem 2. Theorem 2 thus shows that imperfect audits technology promotes the use of deterministic audits. Theorems 1 and 2 are the main results of the paper.

So, introducing imperfect audits encourages deterministic audit regimes. When risk sharing is considered to be of high value, low reports are audited with certainty. When risk sharing is considered to be of low value, even low reports are never audited. The interaction between leverage and costly, imperfect auditing helps underpin the finding that deterministic incentive schemes are generally optimal. Note that leverage and audit probability are similar in that higher leverage increases the expectation and the variance of consumption, as noted above; so too does a decrease in audit probabilities. Audit costs and quality also play an important role in determining optimal leverage. When audit costs are low, optimal leverage is such as to permit large gains from insurance or auditing. This is what Gale and Hellwig (1985) also find in their seminal paper. When audit quality is low, the cost of enforcement increases quickly in leverage, restricting optimal leverage below the perfect audits case (Propositions 4, 5).

1 Literature

Equity finance typically allows issuers to reduce repayments or dividends in bad times whilst reductions in the value of assets are shared between borrowers and lenders. Debt finance is more rigid. Debts are only reduced or discharged in bankruptcy, which follows large falls in income or asset values. So, surely it would be better if there was less debt and more equity?

Townsend (1979) was first to propose an explanation for the prevalence of debt contracts. He shows that when a risk averse borrower's income is costly to verify a standard debt contract is superior either to a strict debt contract, where repayments are constant across states, or a standard equity contract, where repayments are proportional to the borrower's income. The difficulty with the equity contract is that to ensure the borrower does not misreport income the investor needs to undertake a costly audit regardless of the report. A superior contract prescribes audits and risk sharing only following sufficiently low reports, when the borrower's marginal utility and sensitivity to risk are highest. If the borrower's income is sufficiently high, they make a fixed repayment and absorb any remaining income risk at the margin. Such a contract is the standard debt contract that is widespread in personal and corporate loan markets.

Townsend's analysis constrained agents to deterministic auditing regimes. However, he suggested a better contract might employ a stochastic auditing schedule (Townsend, 1979, Section 4): Randomising audits following declarations of default would reduce resource costs while retaining the incentive compatibility of truthful reporting. Using stochastic auditing schemes would allow more risk sharing across states with fewer resources spent on audits across a portfolio of loans. This conjecture was confirmed by Border and Sobel (1987) and Mookherjee and Png (1989). These authors also emphasised the positive role of audits in enforcement; agents should be rewarded following verified truthful reports. In sum, deterministic audits of low reports are unnecessary for contract enforcement, and stochastic reports of moderate and high reports are worthwhile when the costs of audit are low. The resulting optimal contracts resemble equity finance–repayments are contingent on marginal fluctuations in income even across relatively high states.

That risk sharing comes at a cost that is not captured in the benchmark model. In order to ensure truth-telling when the probability of audit is low, audits that contradict the borrower's report can result in penalties far larger than the amount borrowed. If that audit technology were to contradict a *truthful* report, then the prospect of sizeable wrongful penalties might render such contracts unacceptable to the borrower. Indeed, even if the entrepreneur were merely to *fear* that audits may not be perfect, or that their truthful report may be disputed by the lender or bankruptcy court, they would likely balk at a contract that leaves open the prospect of large penalties following disputed reports. In short, equity-like contracts provide more insurance across states, but may exacerbate already bad situations for a borrower. Hence the motivation of this paper.

Perhaps surprisingly, there have been relatively few studies of the impact of imperfect audits.[2] Earlier contributions to CSV problems with audit errors have focused on insurance problems in the context of a risky endowment. Haubrich (1995) presents a model where weakly informative audits are rarely used in efficient contracts. Alary and Gollier (2004) study an example with no commitment to audits, showing that the occurrence of strategic default is dependent on the preferences of the agent. Imperfect signals are also commonly employed in the law enforcement literature. Polinsky and Shavell (2007) is an insightful review of that literature covering amongst other things the impact of risk aversion on optimal penalties. They show that optimal penalties (deterrence) are typically lower for risk averse agents compared to the risk neutral case. A different literature assumes that project outcomes are observable, yet entrepreneurial actions are partially observable. Efficient contracts must encourage entrepreneurs to exert privately costly effort. In these models, the concepts of debt and equity finance are related solely to the optimal sensitivity of repayments to project outcomes. A recent example which rationalizes a combination of debt and equity in this setting with partially observable actions and limited enforceability is Ellingsen and Kristiansen (2011).[3]

Above we noted the link between optimal leverage and audit costs. Gale and Hellwig (1985) also studied the effects of audit costs and risk aversion on leverage in a costly state verification model with perfect and deterministic audits. Our analysis permits stochastic audit regimes, and finds alternative interactions between leverage and the contracting environment: leverage has a dramatic impact on the nature of the efficient contract in our model, and it is the joint determination of leverage and incentive regime which encourages debt contracts in our framework.

1.1 Further features of optimal contracts

As in Townsend's (1979) original analysis, our model motivates an endogenous form of limited liability.[4] Following default, the optimal repayment is determined by both reported income and the revealed audit signal. The severity of repayment following disputed low report varies with the parameterisation of the model: under some parameterisations, as in our numerical example, there is considerable loan relief even following disputed reports.

In our model, when the audit technology is relatively accurate, the costs associated with wrongful penalties and audit errors decline. Optimal contracts involve a high degree of risk

[2] That audits may be less than perfect in practice appears to be widely acknowledged and documented. See Bazerman et al. (2002).

[3] The literature studying the choice between debt and equity and the specifics of those contracts is, of course, large and diverse. One branch of the literature, associated with corporate finance, argues that equity issue is costly as it signals poor quality management; debt issue thus maintains the existing value of equity. Another branch views debt contract structure as a solution to the problem of the entrepreneur/borrower not being able to commit not to withdraw specialist skills. Finally, and particularly in the banking literature, it is often argued that funding should be heavily skewed towards (short-term) debt as a way to discipline bank managers: debt-holders enjoy control rights in low-return states of the world that equity holders do not. Of course, other perspectives also exist. Debt issue may be able to exploit a tax shield not applicable to equity. And debt may be the result of skewed incentives facing managers and shareholders from so-called 'debt overhang'.

[4] In applications of Townsend's framework with risk neutrality, including Gale and Hellwig (1985) and Bernanke, Gertler, and Gilchrist (1999), liability is only limited by the inability to pay; the lender simply takes everything upon default.

sharing, with larger penalties with lower audit probabilities. Essentially, these contracts resemble equity even if the resource costs of audit are significant. In this sense, our framework nests both equity-like contracts and debt-like contracts as optimal under various configurations. In Section 7 we draw on Herranz et al. (2015) amongst others and show that under plausible parameterizations, the model can explain both debt and equity contracts. We show that audit quality is the prime determinant as to which type of contract is optimal. Moreover, the predicted debt contracts generate equilibrium relationships between interest rates, default probabilities and leverage ratios that are broadly consistent with empirical estimates.

In the model of Bernanke, Gertler, and Gilchrist (1999), based largely on Townsend (1979), entrepreneurs are risk neutral. The credit spread between loans and the risk free rate is volatile and responds to marginal fluctuations in the probability of default, which drive the expected resource costs of future auditing. In our model, this credit spread also responds to entrepreneurs' demands for insurance against bad states, which determines the distribution of losses between external creditors and entrepreneurs upon default.

1.2 COMMITMENT

This paper, along with the aforementioned studies, considers an environment where the lender is able to commit ex ante to an incentive regime which is wasteful ex post. That commitment may indicate a concern for reputation, or delegation to a specialised auditor or bankruptcy court as in Melumad and Mookherjee (1989). Krasa and Villamil (2000) investigate what happens when lenders cannot commit to costly audits. That lack of commitment means the revelation principle does not hold; borrowers in equilibrium misreport their income with positive probability. It turns out that lack of commitment means that deterministic audits may be a feature of the optimal contract. Audits can only occur if the expected value of penalties levied following audits exceeds the audit costs. If true for a particular reported income, then this report will be audited with certainty. In short, for Krasa and Villamil (2000) the ability to commit implies equity-like contracts are preferable, whereas for us it does not.

1.3 ROADMAP

The rest of the paper is set out as follows. Section 2 lays out the model environment and the nature of the auditing technology. Section 3 establishes key features of efficient contracts. In section 4 we present the perfect audits benchmark (and Theorem 1). Section 5 explores the imperfect audits case, and establishes that debt contracts may be globally optimal (Theorem 2). Section 6 presents a special case of the model where closed form solutions can be obtained for a global optimum. Section 7 provides numerical analysis of the general model showing that debt is the globally optimal contract when audit quality is low enough. Moreover, the model generates empirically plausible equilibrium relationships between interest spreads, default probabilities and leverage. Section 8 offers concluding remarks. Appendices contain formal arguments and proofs. Figures are contained in Appendix C.

2 THE ENVIRONMENT

We study the one period problem of a risk averse and credit constrained entrepreneur. The assumption of risk aversion is supported by empirical evidence. The most relevant study for our analysis is Herranz et al. (2015), who estimate the median coefficient of relative risk aversion for small business owners in the US Federal Reserve Survey of Small Business Finances of 1.5. It is important to note that the assumption of risk aversion for entrepreneurs does not directly translate into risk averse firm decision making. The entrepreneur's exposure to and appetite for the firm's risk taking is dependent on the risk sharing capacity of their capital structure.

The entrepreneur has access to a special technology offering high returns which are uncorrelated with other projects undertaken in the economy. The outcome of the project is initially private information to the entrepreneur, limiting the sharing of risk between the entrepreneur and a financial intermediary. Contract repayments are enforceable, but can only be conditioned on public information. The public information available to condition contracts includes any message sent by the entrepreneur, m, and any audit signal produced by the audit technology, σ. The entrepreneur makes a take-it-or-leave it contract offer to the financial intermediary, who is well-diversified and operating in a perfectly competitive market. An optimal contract maximises the entrepreneur's expected utility.[5]

2.1 THE ENTREPRENEUR

The entrepreneur enjoys consumption at the end of the period according to $\mathcal{U}(x) : X \to \mathbb{R}$, where $\mathcal{U}', -\mathcal{U}'' > 0$, and X is a closed, right unbounded interval of real numbers.[6] The entrepreneur brings wealth α into the period. Combining the entrepreneur's wealth α with the net funds transferred from the financial intermediary b, the project produces the consumption good according to stochastic gross return $(\alpha + b)\theta$. The revenue shock θ is drawn from a discrete distribution, $\theta \in \Theta$, where Θ is a set of possible distinct values of the shock θ occurring with non-zero probability. By convention, we order the values of Θ as $\{\theta_1, \theta_2, ..., \theta_n\}$, where $\theta_i > \theta_j$ iff $i > j$. The unconditional probability of revenue draw θ_i is denoted $\Delta(\theta_i)$, with $\sum_{i=1}^{n} \Delta(\theta_i) = 1$. By construction, $\Delta(\theta_i) \in (0, 1]$ $\forall i \in \{1, 2, ..., n\}$. The operator $\Delta(\cdot)$ will be used throughout this paper to generate probability measures over its arguments.

Following the realisation of their project, the entrepreneur can send a public signal indicating the state and subsequent revenues of the project. As we study direct truth-telling mechanisms we can restrict the message space as follows: message m is drawn from $M = \{m_1, m_2, ..., m_n\}$, where a message of m_i is interpreted as a declaration that the entrepreneur has received revenue

[5] An alternative formulation would be to maximise the profits of the financial intermediary subject to satisfaction of some participation constraint of the entrepreneur. This distinction has no effect on the main results of this paper.

[6] That is, either $X = [\underline{X}, +\infty)$ for some $\underline{X} \in \mathbb{R}$ or $X = (-\infty, +\infty)$.

shock θ_i. As the revenue shock θ_i is initially only observed by the entrepreneur, entrepreneurs may have an incentive to misreport (that is, to report m_i when the true revenue is θ_j for some $j \neq i$). Under any truth-telling mechanism, equilibrium messaging obeys the following conditional probability distributions $\Delta(m_i|\theta_i) = 1 \ \forall i \in \{1, 2, ..., n\}$. The message m is modeled as a revelation of assets held by the entrepreneur. It is assumed that the entrepreneur can hide assets, but cannot hypothecate assets. Formally, an agent can reveal message m_i if and only if their true revenue shock is greater than or equal to θ_i.[7]

2.2 The financial intermediary

There exists a well-diversified financial intermediary who can make credible commitments to future actions.[8] Any contract involving the entrepreneur and the financial intermediary is small from the perspective of the financial intermediary's balance sheet. The entrepreneur's technology shock θ is uncorrelated with other shocks in the economy, and the returns of other assets/liabilities of the financial intermediary's balance sheet. It follows that the financial intermediary is risk neutral with respect to claims contingent on the entrepreneur's individual specific technology shock θ.

The financial intermediary operates in a perfectly competitive market. Their opportunity cost of funds is given by ρ; any contract offering an expected return on possibly state contingent loans exceeding ρ is acceptable. This condition is formalised below in Constraint 3. The opportunity cost of funds could be thought of as some combination of the interest rate paid by a risk free bond, the interest rate paid by the intermediary to their deposit holders and the intermediary's administrative costs.

The following two assumptions ensure that there are positive, finite gains from trade between the entrepreneur and financial intermediary.

Assumption 1 *Expected project returns exceed the financial intermediary's opportunity cost of funds,* $\sum_{\theta\in\Theta} \Delta(\theta)\theta > \rho$.

Assumption 2 *In the lowest state, project returns are lower than the financial intermediary's opportunity cost of funds,* $\theta_1 < \rho$.

Assumption 1 ensures that there are economic gains from diverting resources to the entrepreneur's project, even when the entrepreneur has access to a deposit facility at the bank yielding a risk free return equal to the bank's opportunity cost of funds, ρ. Assumption 1 is strong enough to ensure that $b > -\alpha$. Assumption 2 specifies that the entrepreneurs' projects are risky. In bad states, a project will yield lower returns than the risk free asset. Assumption 2 is a necessary but not sufficient condition for the existence of finite leverage optimal contracts.

[7] The revelation of assets during the message reporting relaxes the constraint set of the optimal contract problem.

[8] Efficient contracts will typically require commitment on behalf of the financial intermediary. One might think of this as sustained either through the intermediary's concern for its reputation, or through delegation to a specialist bailiff or auditor as in Melumad and Mookherjee (1989).

2.3 AUDITS

There exists an audit technology $\mathcal{T}(\kappa, \mathcal{S}(\theta))$ which is characterised by an audit cost parameter κ and a mapping $\mathcal{S}(\theta)$ from realised revenues θ to distributions of audit signals σ. The resource cost of an audit is the product $\kappa(\alpha + b)$; that is, audit costs are linearly increasing in the assets controlled by the entrepreneur. Following an audit, the audit technology produces a signal $\sigma \sim \mathcal{S}(\theta)$. This signal σ is assumed to be drawn from a discrete set of potential audit signals denoted by Σ. The action to undertake an audit is common knowledge, and so is the signal provided, σ. The entrepreneur knows if (s)he has been audited, and if so the result of the audit. The audit technology is exogenous, we do not allow agents to choose between competing technologies.[9]

The signal produced by the audit technology maps from the space of realised shocks θ as follows: if there is no audit, the audit signal is the empty set, $\sigma = \emptyset$; if there is an audit, the audit signal is drawn from set Σ. The cardinality of the set of possible signals $|\Sigma|$ does not necessarily equal the cardinality of the set of possible revenue outturns $|\Theta|(= n)$. Also, for now, we do not require an ordering of the elements of the set of possible signals, Σ. The probability of revenue shock θ conditional upon audit signal σ is denoted $\Delta(\theta|\sigma)$.

An *audit strategy* is a mapping from messages to audit probabilities and is denoted $\mathcal{Q}: m \to [0, 1]$. Under truth-telling mechanisms, we can restrict our attention to reports $m \in \{m_1, m_2, ..., m_n\}$ and we denote $q_i = \mathcal{Q}(m_i)$. It is assumed that an audit strategy can be agreed and committed to ex ante. Audit strategies are defined in contracts, and implemented ex post by the financial intermediary. The probability of the couplet (σ, θ) conditional on the probability of audit q is denoted $\Delta(\theta, \sigma|q)$.

Definition 1 specifies what is meant by the terms *perfect audits* and *imperfect audits*.

Definition 1 *Imperfect audits and perfect audits:*

a. *Imperfect Audits. An audit technology is* imperfect *if and only if there exists some couplet* $(\theta_i, \sigma) \in \Theta \times \Sigma$ *such that* $\Delta(\theta_i, \sigma|1) > 0$ *and* $\Delta(\theta_i|\sigma) \in (0, 1)$.

b. *Perfect Audits. An audit technology is* perfect *if and only if for all couplets* $(\theta_i, \sigma) \in \Theta \times \Sigma$ *the following holds: if* $\Delta(\theta_i, \sigma|1) > 0$*, then* $\Delta(\theta_i|\sigma) \in \{0, 1\}$.

Note that under perfect audits, it is possible that multiple audit signals perfectly predict a single revenue shock. That is, it is possible that there exist two distinct signals $\sigma, \sigma' \in \Sigma$ with the property that for some θ_i, $\Delta(\theta_i, \sigma|1) > 0$ and $\Delta(\theta_i, \sigma'|1) > 0$ and $\Delta(\theta_i|\sigma) = \Delta(\theta_i|\sigma') = 1$. In this example, the revenue shock θ_i does not predict a unique audit signal with certainty (by Bayes' law, $\Delta(\sigma|\theta_i) \in (0, 1)$). However, the audit signals σ, σ' do predict a specific revenue shock θ_i with certainty. It is the latter that matters for contract enforcement.

[9] This would be an interesting extension of the model. In particular, it may be the case that the optimal auditing technology used to enforce equity contracts differs from that used to enforce debt contracts. This may help us understand the coexistence of debt and equity finance issued by individual firms.

3 CONTRACTS

We have already described two elements of financial contracts: b is the amount of real resources transfered from the financial intermediaries to entrepreneurs at the beginning of the period to invest in the project; $\mathcal{Q}$ is the audit strategy that specifies the probabilities that audits will occur conditional upon messages m announced by the entrepreneur following the private realisation of the revenue shock θ.

The third element of a financial contract is the *repayment function*. The repayment function maps message and audit signal pairs to real transfers of resources from the entrepreneur to the financial intermediary at the end of the period, $\mathcal{Z} : \; M \times \sigma \to \mathbb{R}$. This repayment does not need to be strictly positive. Under truth-telling direct mechanisms, we denote $\mathcal{Z}(m_i, \sigma)$ by $z_i(\sigma)$. The fourth element of the financial contract is the consumption allocation function. The consumption function maps the revenue state, the message and the audit signal to final consumption of the entrepreneur $\mathcal{X} : \; \Theta \times M \times \sigma \to \mathbb{R}$. The consumption allocation function is denoted by $\mathcal{X}(\theta, m, \sigma)$, where the ordering of the tuple (θ, m, σ) replicates the timing of the model; first the entrepreneur receives revenue shock θ, then reports message m, before the audit signal σ is revealed. We will typically focus our analysis on audit strategies $\mathcal{Q}$, borrowing b and repayment allocations $\mathcal{Z}$; the consumption allocations $\mathcal{X}$ are uniquely determined by the repayment allocations and borrowing by the budget constraint (1).

Definition 2 *A* contract *is a tuple $\mathcal{C} = (b, \mathcal{Q}, \mathcal{Z}, \mathcal{X})$ which is agreed at time zero and is common knowledge. A contract is a combination of an amount of resources transfered from the financial intermediary to the entrepreneur for investment, b, an audit strategy, $\mathcal{Q}$, a repayment function $\mathcal{Z}$ and a consumption allocation function $\mathcal{X}$.*

The motivation for this paper is the search for environments where optimal contracts resemble standard debt contracts, which we define as follows.

Definition 3 *We specify the following two benchmark classes of debt contracts.*

a. A non-contingent debt *contract is a contract with constant repayments across all states and messages* $z_i(\sigma) = z_j(\sigma') \quad \forall m_i, m_j \in M, \; \sigma, \sigma' \in \Sigma$.

b. A standard debt *contract has the following two properties: (1) the contract specifies a constant repayment when either the entrepreneur's message is equal to or above some threshold* m_k*,*

$$z_i(\sigma) = z_j(\sigma') \quad \forall m_i, m_j \in \{m_k, m_{k+1}, ..., m_n\}, \; \sigma, \sigma' \in \Sigma;$$

(2) all low reports are audited, $q_i = 1 \; \forall i < k$.

We refer to the reporting of a message m_i *where* $i < k$ *is interpreted as* default.

Note that debt contracts in our model do not restrict the entrepreneur borrower to zero consumption following default. In fact, in the examples that we consider, entrepreneurs will enjoy strictly positive consumption in all circumstances, even following a default. This positive consumption could represent income already paid to the entrepreneur during the life of the project.

3.1 Constraints

Contracts in our framework are subject to four classes of constraints. The first class of constraints are budget constraints. The budget constraints are equality constraints in our problem: any individual entrepreneur's consumption can neither exceed ($\leq$) nor fall short of ($\geq$) the difference between revenue $(\alpha+b)\theta$ and repayments $\mathcal{Z}(m,\sigma)$. Typically, optimal contracts will be constrained by their inability to increase the consumption of truth-telling agents above the level specified by the budget constraints, and will be constrained by their inability to decrease the consumption of misreporting agents below the level specified by the budget constraints. The budget constraints are present regardless of assumptions about the behaviour of the financial intermediary or the restriction to truth-telling mechanisms.

Constraint 1 *Budget constraints. State contingent budget constraints are specified as follows:*

$$\mathcal{X}(\theta_i, m_j, \sigma) = (\alpha + b)\theta_i - \mathcal{Z}(m_j, \sigma) \qquad \forall (m_i, \sigma, \theta_j) \in M \times \Sigma \times \Theta. \tag{1}$$

We will typically refer to equation 1 as $\mathrm{BC}_{i,j,\sigma}$. The left hand side of Equation 1 is the revenue received by the entrepreneur from their project. Following the repayment $\mathcal{Z}(m,\sigma)$, the remainder available for the entrepreneur to consume is $\mathcal{X}(\theta,m,\sigma)$. We will frequently use the short hand notation $\mathcal{U}(\theta,m,\sigma) := U(\mathcal{X}(\theta,m,\sigma))$ and $\mathcal{U}'(\theta,m,\sigma) := U'(\mathcal{X}(\theta,m,\sigma))$.

The second class of constraints is the set of bounds on audit probabilities.

Constraint 2 *Audit probability constraints.*

$$\mathcal{Q}(m) \geq 0 \quad \forall m \in M. \tag{2}$$

$$1 - \mathcal{Q}(m) \geq 0 \quad \forall m \in M. \tag{3}$$

The third class of constraints is the participation constraint. By assumption, a contract is *acceptable* to the financial intermediary if and only if the expected repayment exceeds the sum of the opportunity cost of funds and any audit costs incurred.

Constraint 3 *Participation Constraint. The participation constraint is specified as follows:*

$$\sum_{m \in M} \Delta_f(m) \sum_{\sigma} \Delta_f(\sigma | m, \mathcal{Q}(m)) \mathcal{Z}(m,\sigma) \geq b\rho + \sum_{m \in M} \Delta_f(m)\mathcal{Q}(m)(\alpha + b)\kappa, \tag{4}$$

where the probability measures $\Delta_f(\cdot)$ are constructed from the financial intermediary's information set.

The left hand side of equation 4 captures the expected repayment constructed from the financial intermediary's information set Ω_f. Under any contract, the financial intermediary must forecast the entrepreneurs messaging strategy to form an expectation of repayments. The revelation principle holds in our setting. This means that there exists an optimal contract under which the entrepreneur weakly prefers truthfully to reveal their true return θ in all states. We refer to contracts that induce truth-telling as *truth-telling contracts*. Under any truth-telling contract, $\Delta(m_i|\theta_i) = 1 \ \forall i \in \{1, 2, ..., n\}$. Therefore, under truth-telling we can re-write Constraint 3 as follows:

$$\sum_{\sigma} \Delta(\theta_i, \sigma|q_i) z_i(\sigma) - b\rho - \sum_{i=1}^{n} \Delta(\theta_i) q_i(\alpha + b)\kappa \geq 0. \tag{5}$$

We will typically refer to equation 5 as PC.

The fourth class of constraints is the set of incentive compatibility constraints. The incentive constraints ensure that for entrepreneurs, a strategy of always truthfully revealing their true return weakly dominates all other reporting strategies. This incentive compatibility constraint is formalised by Constraint 4.

Constraint 4 *Incentive compatibility constraints. Contracts are referred to as* incentive compatible *if and only if the following constraints hold:*

$$m_i \in \arg\max_{m} \sum_{\sigma} \Delta(\theta_i, \sigma|\mathcal{Q}(m))\mathcal{U}(\theta_i, m, \sigma) \quad \forall i \in \{1, 2, ..., n\} \tag{6}$$

It is useful to deconstruct the Incentive Compatibility Constraint into a set of constraints comparing individual pairs of reports. That is, a contract is incentive compatible if for all state pairs (θ_i, θ_j), an entrepreneur receiving true return θ_i weakly prefers to report m_i over m_j. This pairwise formulation of the Incentive Compatibility Constraint is equivalent to 4 and is formalised by 7:

$$\sum_{\sigma} \Delta(\theta_i, \sigma|q_i)\mathcal{U}(\theta_i, m_i, \sigma) - \sum_{\sigma} \Delta(\theta_i, \sigma|q_j)\mathcal{U}(\theta_i, m_j, \sigma) \geq 0 \quad \forall i \in \{1, 2, ..., n\}, \ j < i. \tag{7}$$

We will typically refer to equation 7 as $\text{ICC}_{i,j}$ for truthful report m_i and misreport m_j. There are two challenges to enforcement in the model under imperfect audits. First, penalties may be wrongfully applied to truth-telling agents. Second, even when the audit signal has identified a fraudulent report with certainty, the financial intermediary may remain uncertain of the true income of the misreporting agent, and may therefore be unable to impose a maximal penalty. Remark 1 ensures that all penalty repayment allocations are payable by any agent who may be charged this allocation. Remark 1 becomes redundant when the domain of the utility function is unbounded below (that is, when $\text{Dom}(\mathcal{U}) = \mathbb{R}$), in which case any agent is able to pay any repayment regardless of assets.

Remark 1 *For Constraint 4 to be well-defined, it must be the case that consumption allocations both on and off the equilibrium path are in the domain of the utility function:*

$$\mathcal{X}(\theta_i, m_j, \sigma) \in X \quad \forall\{(i, j, \sigma) \mid \Delta(\theta_i, \sigma|q_j) > 0 \ \wedge \ j \leq i\}.$$

Before continuing, we define the concept of a feasible contract, and the active set of constraints.

Definition 4 *A contract $\mathcal{C}$ is* feasible *if and only if the constraints $BC_{i,j,\sigma}, PC, ICC_{i,j}$ are satisfied for all i, j, σ. Let $\mathcal{C}$ be a feasible contract. The* active set *of constraints, denoted $\mathcal{A}(\mathcal{C})$, is the set of binding constraints,*

$$BC_{i,j,\sigma} \in \mathcal{A}(\mathcal{C}),$$

$$PC \in \mathcal{A}(\mathcal{C}) \iff PC = 0,$$

$$ICC_{i,j} \in \mathcal{A}(\mathcal{C}) \iff ICC_{i,j} = 0.$$

3.2 Optimal contracts

Optimal contracts are formalised by Programme 1. We assume that the optimal contract maximises entrepreneurs' utility conditional upon participation of the financial intermediary and truth-telling. Note that there may be welfare maximizing contracts that are not classed as optimal under this definition, if these contracts do not induce truth-telling as a dominant reporting strategy. By the revelation principle, we know that there exists a welfare maximizing contract that does induce truth-telling, and our definition of optimality restricts our attention to these truth-telling contracts.

Programme 1 *A contract is* optimal *if and only if it maximises the entrepreneur's utility subject to feasibility*

$$\max_{\mathcal{C}} \sum_{i,\sigma} \Delta(\theta_i, \sigma|q_i)\mathcal{U}(\theta_i, m_i, \sigma) \tag{8}$$

subject to

$$(1),\ (2),\ (3),\ (5),\ (7).$$

The first order conditions of Programme 1 will not always provide sufficient, or even necessary conditions for optimal contracts. Sufficiency is broken as a result of the non-convexity of the constraint set. The necessity of the first order conditions of Programme 1 is also not guaranteed; when there exists some i such that $q_i \in \{0, 1\}$, there can exist optimal contracts that do not satisfy the first order condition for audit probability q_i. These results are formalised by Remark 2.

Remark 2 *The First Order Conditions of Programme 1 do not specify sufficient nor necessary conditions for optimal contracts.*

The proof of Remark 2 along with the proofs of all other results are contained in Appendix A. Non-sufficiency is shown by Proposition 7, which states that some parameterisations permit multiple locally optimal contracts. Constraints 1 and 2 are affine in the choice variables $\mathcal{Z}, \mathcal{Q}, \mathcal{X}, b$. Constraints 3 and 4 are non-convex. First, both Constraints 3 and 4 contain products of probabilities with repayments and utility allocations, all of which are choice variables and some of which enter into the constraint with negative signs. Second, the second term in Equation 7 includes lotteries over utility allocations. In terms of the choice variable consumption x, the expected value of these lotteries is convex, but the negation of this expectation is non-convex. In response to this second problem, some authors consider utility allocations rather than consumption allocations as the choice variable. This change in choice variable would not convexify our problem, as both Constraints 3 and 4 would still include products of choice variables entering into the constraints with both positive and negative signs. This non-convexity is material for our analysis.[10]

Proposition 1 *Let $\mathcal{C}$ be a globally optimal contract. $\mathcal{C}$ satisfies the following necessary conditions.*

Repayments $\mathcal{Z}(m_i, \sigma)$ satisfy

$$\frac{\mathcal{U}'(\theta_i, m_i, \sigma)}{\lambda} = \frac{1}{1 + \sum_k \mu_{i,k}} \left[1 + \sum_j \mu_{j,i} \frac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)} \frac{\mathcal{U}'(\theta_j, m_i, \sigma)}{\lambda} \right] \tag{9}$$

for all (m_i, σ). The initial transfer of resources to the entrepreneur b satisfies

$$\sum_i \Delta(\theta_i)[\theta_i - q(\theta_i)\kappa] - \rho = \sum_{i,\sigma} \Delta(\theta_i, \sigma|q_i) \left[\sum_j \mu_{j,i} \frac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)} \frac{\mathcal{U}'(\theta_j, m_i, \sigma)}{\lambda} (\theta_j - \theta_i) \right]. \tag{10}$$

for all $i \in \{1, 2, ..., n\}$, and

$$\lambda > 0, \quad \mu_{j,i}, \geq 0 \; \forall \; i, j, k.$$

Equation 9 presents the optimality condition for repayment $z_i(\sigma)$. The left hand side is the ratio of the shadow cost of the participation constraint, λ, and the marginal utility of consumption in state (θ_i, σ). This ratio is a marginal rate of substitution from expected consumption

[10] Furthermore, in contrast with Grossman and Hart (1983) and Bolton (1987), re-writing the problem in terms of utility allocations does not convexify the sub-problem of optimising repayment allocations conditional upon a given audit strategy. To see this, consider the substitution $\mathcal{X}(\theta, m, \sigma) = \xi(\mathcal{U}(\theta, m, \sigma))$, where $\xi = \mathcal{U}^{-1}(\mathcal{X}(\theta, m, \sigma))$. The function ξ is convex, and after substitution appears in the budget constraints (1), which become non-convex. In Programme 1, the budget constraints are equality constraints: they both constrain the consumption of truth telling agents from above, and constrain the punishments imposed on misreporting agents from below. The introduction of any non-linearity into the budget constraints breaks the convexity of these constraints.

marginal utility to realised consumption marginal utility and is equal to one under perfect information. This marginal rate of substitution is decreasing in $\sum_j \mu_{i,j}$, the sum of shadow costs of binding incentive constraints in which the repayment $z_i(\sigma)$ enters in the first term of the constraint (7). In other words, this term captures the cost of ensuring that an agent earning θ_i reports m_i truthfully. All else equal, this cost of ensuring truthful reporting is high when the marginal utility of consumption is low relative to expected marginal utility.

The term $\sum_j \mu_{j,i} \dfrac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)} \dfrac{\mathcal{U}'(\theta_j, m_i, \sigma)}{\lambda}$ captures the cost of ensuring that agents earning θ_j do not report m_i. The sum $\sum_j \mu_{j,i}$ is not dependent on σ and captures the shadow costs of binding incentive constraints in which repayments $z_i(\cdot)$ enter in the second term of the constraint (7). Conditional upon the signal σ these incentive costs are increasing in the marginal likelihood ratio $\dfrac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)}$. When this marginal likelihood ratio is close to unity, the signal σ leaves the lender unable to detect misreporting by agent earning θ_j and reporting m_i. The final term, $\dfrac{\mathcal{U}'(\theta_j, m_i, \sigma)}{\lambda}$, captures the marginal increase in utility of a misreporting agent receiving θ_j and reporting m_i, conditional upon the audit signal σ. This marginal utility is normalised by the shadow cost of the participation constraint, λ. All else equal, when this marginal utility is low, a marginal increase in $z_i(\sigma)$ has a small increase in the value of misreporting m_i for an agent receiving θ_i, and therefore a small impact on incentive compatibility.

Equation 10 captures the optimality condition for the initial amount of resources transferred from the financial intermediary to the entrepreneur, b. The left hand side captures the net contribution of a marginal increase in b to expected consumption. The first term $\sum_i \Delta(\theta_i)\theta_i$ is the expected per unit increase in revenue. The second term $\sum_i \Delta(\theta_i)q(\theta_i)\kappa$ is the expected increase in audit costs and ρ is the financial intermediary's opportunity cost of funds. The right hand side captures the incentive costs of an increase in b. The term $\sum_{i,\sigma} \Delta(\theta_i, \sigma|q_i)$ takes the expectation over states (θ_i, σ), conditional upon the audit strategy $\mathcal{Q}(m)$. The sum $\sum_j \mu_{j,i}$ captures the shadow costs of the binding incentive constraints relating to entrepreneurs receiving revenue θ_j who weakly prefer reporting m_j over m_i. These shadow costs are scaled by three factors: First, the marginal likelihood ratio $\dfrac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)}$ captures the ability of the lender to identify misreporting agents earning θ_j and reporting m_i; second, the term $\dfrac{\mathcal{U}'(\theta_j, m_i, \sigma)}{\lambda}$ captures the marginal increase in the value of an agent receiving θ_j and misreporting m_i conditional upon audit signal σ; third, the term $(\theta_j - \theta_i)$ captures the rate at which a rise in borrowing increases the absolute risk across revenue states.

Corollary 1 *Let $\mathcal{C}$ be a globally optimal contract. The repayment terms of contract $\mathcal{C}$ satisfy the following conditions:*

a. A repayment allocation $\mathcal{Z}(m_i, \sigma)$ is contingent on the audit signal σ only if the message m_i enters on the right hand side of a binding incentive constraint. That is,

$$\exists\, \sigma, \sigma' \text{ s.t. } \mathcal{Z}(m_i, \sigma) \neq \mathcal{Z}(m_i, \sigma') \implies \exists\, \theta_j \text{ s.t. } \text{ICC}_{j,i} \in \mathcal{A}(\mathcal{C}).$$

b. Under non-increasing absolute risk averson, if the incentive constraint $ICC_{i,j}$ is binding, then repayments are strictly increasing in the conditional likelihood ratio of θ_j with respect to θ_i. That is,

$$A'(x) \leq 0 \wedge ICC_{j,i} \in \mathcal{A}(\mathcal{C}) \wedge \frac{\Delta(\theta_j|\sigma')}{\Delta(\theta_i|\sigma')} > \frac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)} \implies \mathcal{Z}(m_i, \sigma') > \mathcal{Z}(m_i, \sigma),$$

where $A(x) := \dfrac{-\mathcal{U}''(x)}{\mathcal{U}'(x)}$ is the Arrow-Pratt measure of absolute risk aversion.

Corollary 1 formalises two properties of optimal repayments. First, any repayment that only appears on the left hand side of binding incentive constraints—this includes the highest repayment θ_n, is not contingent on audit signals. Increases in the uncertainty of consumption following report i decreases expected utility for agents earning θ_i and also decreases the incentive for these agents truthfully to report income. These costs can be compensated if the increased uncertainty deters false reports from agents with true income greater than i. But, if all incentive constraints ICC$_{j,i}$ are non-binding, then there is no benefit attainable from uncertainty in repayments following report m_i. Second, if a repayment allocation is present on the right hand side of a binding incentive constraint, repayments are increasing in the conditional likelihood ratio of misreporting j with respect to truthful reporting i. The conditional likelihood ratio is the marginal rate of transformation of consumption for misreporting agents with respect to truth telling agents. When the conditional likelihood ratio is high, the disincentive effect of high repayments is large relative to the direct effect on expected utility.

Proposition 2 *Let $\mathcal{C} = (\mathcal{Q}, \mathcal{Z}, \mathcal{X}, b)$ be a globally optimal contract. Contract $\mathcal{C}$ has the following properties:*

a. The financial intermediary's participation constraint is binding, PC $\in \mathcal{A}(\mathcal{C})$.

b. The highest possible report is never audited, $\mathcal{Q}(m_n) = 0$

c. There exists a binding incentive constraint, $\exists\, i, j$ s.t. ICC$_{i,j} \in \mathcal{A}(\mathcal{C})$.

Let the audit strategy $\mathcal{Q}^*(m)$ be taken as given, and the probabilities of all revenue states be positive conditional upon any audit signal $(\Delta(\theta_i|\sigma) > 0\ \forall\ \theta_i, \sigma)$.

d. Conditional upon $\mathcal{Q}^*$, under the optimal allocation of repayments $\mathcal{Z}$ and borrowing b there exists a binding incentive constraint $\exists\, i, j$ s.t. ICC$_{i,j} \in \mathcal{A}(\mathcal{C})$.

Proposition 2 characterises restrictions on the active set of constraints under optimal contracts. Part (a) states that the financial intermediary's participation constraint is binding; it is never in the interest of the entrepreneur to enter a contract with expected repayments exceeding the financial intermediary's cost of funds gross of monitoring costs. Part (b) states that audits of the highest reports are always wasteful, this follows from Corollary 1(a). Part (c) states that any optimal contract features a binding incentive constraint; any deviation from this would mean that costly audits are being undertaken wastefully. Part (d) goes further. Under the assumption that any message, audit signal pair could be consistent with truth-telling, there exists a binding incentive constraint when repayments and leverage are chosen optimally, regardless of the given audit strategy. The condition $(\Delta(\theta_i|\sigma) > 0 \; \forall \, \theta_i, \sigma)$ is a stronger condition than imperfect audits, and severely restricts the ability of the financial intermediary to detect misreporting agents.

4 Perfect Audits

In the introduction we stated that the interaction between leverage and costly, *imperfect* audits underpins the optimality of deterministic contracts. Before establishing that and other results it is insightful to analyse the case of *perfect* audits. Theorem 1 restates Mookherjee and Png (1989)'s finding that debt contracts are not optimal. Moreover, we go on to show that optimal leverage is unbounded for reasonable parameter values.

Theorem 1 *(Mookherjee and Png, 1989, Proposition 1) Under perfect audits* $(\delta(\theta_i|\sigma_k) \in \{0,1\} \; \forall \; i,k)$*, any optimal contract without certain immiseration following truthful reports* $(\sum_\sigma \Delta(\theta_i, \sigma|q_i)\mathcal{U}(\theta_i, m_i, \sigma) > \underline{\mathcal{U}} \; \forall \, i)$ *cannot include certain auditing of any report. That is,* $\mathcal{Q}(\theta) < 1 \; \forall \, \theta$*.*

We provide a new proof of Theorem 1. The proof works by starting with a contract with certain auditing, $\mathcal{Q}(\theta_i) = 1$ for some i. We show that the marginal audit following report m_i can be replaced by a lottery that replicates the repayments following audits for truth-telling agents receiving true return θ_i, while retaining incentive compatibility and relaxing the lender's participation constraint. Our proof shows the value of weak audit signals and provides some intuition over their role in promoting standard debt contracts. Under perfect audits, an agent forced to audit with certainty conditional upon a given report m_i would wish to reduce costs by weakening the quality of the audit signal, even via a naïve strategy of introducing a lottery between the true audit signal and an uninformative signal.

We have shown by Theorem 1 that for any fixed level of audit costs, optimal contracts under perfect audits do not apply certain audits following any report $(q_i^* < 1)$. Proposition 3 explores how this result relates to audit costs. As audit costs fall, we would expect optimal audit probabilities to increase. Proposition 3 states that even as audit costs approach zero, optimal audit probabilities approach strictly interior values. It follows that certain audits and standard debt contracts cannot be explained by low audit costs.

Proposition 3 *Under perfect audits, as audit costs approach zero, optimal audit probabilities approach values strictly less than one:* $\lim_{\kappa \to 0+} q_i^* \in [0, 1)$*.*

Proposition 3 shows that the cost of contract enforcement under perfect audits is low. Within costly state verification models, leverage acts as a substitute for reductions in auditing: an increase in leverage increases both the expected consumption and the consumption risk of agents. Proposition 4 shows that the importance of this substitutability for optimal contracts.

Proposition 4 *If audits are perfect* $(\delta(\theta_i|\sigma_k) \in \{0,1\} \;\forall\; i,k)$ *and sufficiently inexpensive* $\kappa < \mathbb{E}(\theta) - \rho$*, and projects enjoy constant returns to scale, optimal leverage and entrepreneur consumption are infinite.*

Estimates of the direct costs of auditing range between 0.01 and 0.06 of assets.[11] While it is not straightforward to estimate the marginal return on assets for entrepreneurs, we can obtain estimates of the average return on assets for entrepreneurs: Herranz et al. (2015) find a mean annual gross return on assets of 1.30 for small firms in the 1993 Survey of Small Business Finances, and a median annual gross return on assets of 1.09 for the same sample. They also estimate the lenders' annualised gross opportunity cost of funds at 1.012. Our model predicts that these values are not consistent with optimal leverage under perfect audits.

5 Imperfect audits

The previous section showed that under perfect audits, it is never optimal to audit a single report m_i with certainty (Theorem 1). In proving this result, we constructed a welfare improving perturbation that bundled the resource-costly perfect audit signal with a lottery; this bundle can be thought of as a lower cost but imperfect audit technology. Under the perturbed contract, the new imperfect audit technology is applied with certainty following report m_i.

Indeed, under imperfect audits, certain auditing of an individual report can be optimal. Optimal contracts can even take the form of standard debt (Theorem 2). Even if the audit probability is high, any further increase in the audit probability does increase the set of feasible consumption allocations available to the entrepreneur. Under imperfect audits, increasing the probability of audit defrays the incentive costs of contract enforcement more widely. This increases risk sharing across states.

Theorem 2 *There exist costly state verification programmes with imperfect audits where optimal contracts take the form of standard debt contracts.*

Theorem 2 shows that standard debt can be optimal under imperfect audits. Standard debt contracts combine the certain auditing of low reported revenues with no audits of high reported revenues. Certain auditing of report m_j requires that the following incentive constraint adapted from (7) must be binding for some θ_i:

[11] See for example Warner (1977), Weiss (1990) and Altman (1984).

$$\sum_{\sigma} \Delta(\theta_i, \sigma | q_i) \mathcal{U}(\theta_i, m_i, \sigma) \geq \sum_{\sigma} \Delta(\theta_i, \sigma | q_j = 1) \mathcal{U}(\theta_i, m_j, \sigma). \tag{11}$$

Under perfect auditing, the signal σ reveals the true income θ_i with certainty. The sum on the right hand side of (11) becomes a single value $\mathcal{U}(\theta_i, m_j, \sigma = \theta_i)$ and this value can be set equal to or arbitrarily close to the lower support of the range of the utility function $\mathcal{U}$ without harming any truth-telling agent. This means that the incentive constraint will be slack for any allocation that does not immiserate with certainty agents who truthfully report m_i.[12]

Under imperfect auditing, the lottery on the right hand side of (11) remains, and repayment allocations $\mathcal{Z}(m_j, \sigma)$ may affect both truth-telling and misreporting agents. Utility allocations on the right hand side $\mathcal{U}(\theta_i, m_j, \sigma)$ are bounded below by what is acceptable ex ante to entrepreneurs who know they may be wrongfully punished for misreporting ex post. In addition, repayments are bounded by the inability of the audit technology to reveal the true income of misreporting agents even when the audit signal can detect whether or not the agent has misreported their income.[13]

This explains how the marginal value of audits can remain positive as the probability of audit approaches one. At the same time, lower quality audits increase the costs of enforcing risk sharing contracts. We show in our proof contained in Appendix A and in the numerical simulations studied in Section 7 that optimal contracts under imperfect audits reduce the probability of audit to zero across high revenue states, where the value of risk sharing is low; optimal contracts increase the probability of audit to one across low revenue states, where the value of risk sharing is high.

We saw in Section 4 that when audits are perfect and sufficiently inexpensive, the optimal leverage ratio is infinite. Consider Equation 10, the first order necessary condition for borrowing b. The right hand side of the equation captures the incentive costs of increasing leverage. When audits are perfect, sufficiently high audit probabilities will leave all incentive constraints slack, and all Kuhn-Tucker multipliers $\mu_{j,i} = 0$. When audits are imperfect, deterministic audit strategies do not guarantee slack incentive constraints. Proposition 5 provides sufficient conditions under which optimal leverage is bounded above:

Proposition 5 *Let the probabilities of all revenue states be positive conditional upon any audit signal,* $(\Delta(\theta_i|\sigma) > 0,\ \forall\ \theta_i, \sigma)$*. Non-negative consumption in all truth-telling states requires*

[12] It is possible to make Equation 11 by introducing an exogenous limited liability constraint. This effectively imposes a lower bound on the support of the range of $\mathcal{U}$. However, for the incentive constraint to bind with certain auditing, the consumption allocations of truth-telling agents would also need to be constrained by the limited liability constraint.

[13] In the example we use to prove Theorem 2, the first channel is important: entrepreneurs write contracts that limit ex post penalties as these penalties are wrongfully applied to truth-tellers with positive probability. In the numerical example we present in Section 7, the second channel is important: the audit technology can correctly identify misreporting agents with positive probability. The optimal standard debt contracts do not need to punish truth-tellers, but they are unable to impose maximal penalties on high income misreporting agents as the audit technology does not precisely reveal the true income of misreporting agents.

that leverage is bounded above as follows:

$$\frac{\alpha + b}{\alpha} \leq \frac{\rho}{\rho - (\theta_1 - \sum_i \Delta(\theta_i)\mathcal{Q}(\theta_i)\kappa)} \tag{12}$$

Proposition 5 shows the effect of imperfect audits on optimal leverage. Under imperfect audits, the possibility of low outturns has an important effect on the limits of leverage. As leverage increases, revenue in the low state approaches zero by Assumption 2. Positive probabilities of revenue states conditional upon any audit signal is a stronger condition than imperfect audits, and implies that any 'penalty' repayment allocation is occasionally wrongfully applied to truth telling agents under contracts with positive probability auditing.

6 The 2-State Model

In the preceding sections we characterised the optimal repayment and borrowing allocations in the general costly-state-verification model with noisy audit signals. Now, we consider a special case of the model with two revenue states. Within the two-state model, we are able to characterise the global optimality of standard debt and describe closed form solutions of the model.

We also simplify the notation. Let $\theta \in \{\underline{\theta}, \bar{\theta}\}$ with $\bar{\theta} > \underline{\theta}$ and $\bar{\pi} = \delta(\bar{\theta}), \underline{\pi} = \delta(\underline{\theta})$. Following an audit, the audit signal is drawn from $\sigma \in \{\underline{\sigma}, \bar{\sigma}\}$ where $\dfrac{\Delta(\bar{\theta}|\bar{\sigma})}{\Delta(\underline{\theta}|\bar{\sigma})} > \dfrac{\Delta(\bar{\theta}|\underline{\sigma})}{\Delta(\underline{\theta}|\underline{\sigma})}$. We define Type-I and Type-II errors as follows:

Definition 5 *A Type-I error occurs when the audit signal is $\bar{\sigma}$ following true income $\underline{\theta}$. A Type-II error occurs when the audit signal is $\underline{\sigma}$ following true income $\bar{\theta}$.*

We denote the conditional probability of Type-I error by $\eta(\underline{\theta}) := \Delta(\bar{\sigma}|\underline{\theta})$. The probability of Type-II error is denoted $\eta(\bar{\theta}) := \Delta(\underline{\sigma}|\bar{\theta})$. Under imperfect audits, either $\eta(\underline{\theta}) > 0$ or $\eta(\bar{\theta}) > 0$ or both.

We proceed allowing borrowing b to be chosen freely, under the assumption of constant technological returns to scale. This does not mean that firms enjoy constant returns to scale. Firm size is endogenously bounded above according to proposition 5. It does mean that the only source of decreasing returns to scale is the information asymmetry between the entrepreneur and external finance providers.

Figure 1 presents evidence of the quantitative relevance of standard debt in our framework for a sample parameterisation. Along the horizontal axis, the risk of the entrepreneur's project $(\bar{\theta} - \underline{\theta})$ is increasing, holding expected returns $(\mathbb{E}(\theta))$ constant. The vertical axis plots audit costs as a share of total assets under management. Two features of the simulation are striking: First, standard debt $(q^* = 1)$ is very prevalent. Very low project risk or high audit costs are required for standard debt to be inefficient. Second, stochastic audit regimes $(0 < q^* < 1)$ are rare. Indeed, when risk is low, efficient contracts 'jump' from standard debt $(q^* = 1)$ to non-contingent debt contracts ($q^* = 0$). One finds that the stochastic audit regimes, which are optimal under the perfect audits framework, are typically rejected in favor of deterministic regimes; entrepreneurs often prefer 'simple', non-contingent or standard debt contracts.

When contracts are constrained by the upper and lower bound on audits ($q^* = 0$ *or* 1), analysis of the entrepreneur's problem is relatively straightforward, yielding closed form solutions under constant relative risk aversion when the likelihood of Type-II audit error is zero ($\eta(\bar{\theta}) = 0$):[14]

Proposition 6 *When the likelihood of Type-I and Type-II errors are positive and zero respectively ($\eta(\underline{\theta}) > 0, \eta(\bar{\theta}) = 0$) and preferences exhibit constant relative risk aversion $U(x) = x^{1-\gamma}/(1-\gamma)$, leverage, consumption allocations and shadow prices of standard debt contracts and non-contingent debt contracts can be represented by closed-form expressions in terms of exogenous parameters.*

The proof of proposition 6 is given in appendix A. Displayed below are solutions to standard debt contracts under logarithmic utility ($\mathcal{U}(x) = \log x$).

$$\begin{aligned} \mathcal{X}(\bar{\theta}, \bar{m}, \emptyset) &= \alpha\rho \, \frac{1}{1-\zeta}, \\ \mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma}) &= \alpha\rho, \\ \mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) &= \alpha\rho \, \frac{\underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}, \\ b &= \frac{\alpha\rho}{\bar{\theta} - \underline{\theta}} \, \frac{\zeta}{1-\zeta} \left(\frac{\bar{\pi} + \underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})} \right) - \alpha, \\ \text{where} \qquad \zeta &= \frac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\bar{\pi}(\bar{\theta} - \underline{\theta})}. \end{aligned} \tag{13}$$

From the solutions presented in (13), we can derive measures of leverage and loan coupon rates. These measures are more easily observed than entrepreneurs' consumption allocations in practise. Leverage, l, as measured by the total assets managed by the entrepreneur over their initial net worth can be described as follows:

$$l = \frac{\alpha + b}{\alpha} = \frac{\rho}{\bar{\theta} - \underline{\theta}} \, \frac{\zeta}{1-\zeta} \left(\frac{\bar{\pi} + \underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})} \right). \tag{14}$$

We can determine the net interest (coupon) rate on loans, r, by subtracting one from the ratio of the repayment following high reports $\mathcal{Z}(\bar{m}, \emptyset)$ and the initial amount borrowed b. Combining

[14]Type-II errors, while perhaps more familiar than Type-I errors, have little effect on the nature of efficient contracts if they occur with a low probability. The implications of Type-II errors are investigated in appendix B.

the budget constraints with (13) yields

$$r = \frac{\mathcal{Z}(\bar{m}, \emptyset)}{b} - 1 = \rho - 1 + \frac{\alpha + b}{b}\underline{\pi}\kappa + \left[\frac{\alpha + b}{b}(\bar{\theta} - \mathbb{E}(\theta)) - (\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset) - \mathbb{E}[\mathcal{X}(\theta, m, \sigma)])\right]. \tag{15}$$

The first term on the right hand side, $\rho - 1$, is the opportunity cost of funds for the financial intermediary expressed as a net interest rate. The second term captures the audit cost component of the interest rate spread. The third term captures compensation to the intermediary for the acceptance of a low repayment following verified low reports. Section 6 presents example comparative statics for leverage ratios and the loan interest rate.

The solutions obtained by proposition 6 are necessary but not sufficient conditions of globally optimal contracts. In the following paragraphs, we consider one interesting example of multiplicity, where programmes exhibit both non-contingent and standard debt contracts as locally optimal, and small changes in parameter values can cause a jumps in the globally optimal contract terms. We formalise this, bang-bang, feature of optimal contracts in Proposition 7, and present computed examples in Figures 1 and 2.

Proposition 7 *When audits are imperfect, there exist parameter specifications which permit both non-contingent* $(q = 0)$ *and standard* $(q = 1)$ *debt contracts as locally optimal contracts.*

When project risk and audit costs are low, optimal contracts can jump from non-contingent debt to standard debt. Figure 2 plots an example of this bang-bang behaviour. In figure 2, the determination of efficient contracts is deconstructed by leverage and audit strategy for one example parameter specification. The horizontal axis plots levels of borrowing. The lower panel plots the efficient probability of audit, conditional upon borrowing. The upper panel plots the attainable expected welfare conditional upon borrowing. The solid line plots expected welfare attainable with a non-contingent contract $(q = 0)$. The dashed line plots expected welfare attainable allowing the audit strategy to be chosen optimally.

The efficiency of standard debt is sensitive to the assumption that the entrepreneur can determine the scale of the project. When leverage is low, total risk is low, and the gains from insurance provided by auditing are low. On the other hand, when leverage is high, auditing will push the minimum consumption allocation closer to zero. Audits are only useful to the extent that the entrepreneur can absorb Type-I errors in low states.

The proof of Proposition 7 solves an example where non-contingent and standard debt contracts are locally optimal. Under the non-contingent debt contract, the marginal resource cost of additional audits exceeds the gains obtained from the audit signal. Under the standard debt contract, the marginal resource gain from reducing the audit probability is smaller than the cost of foregoing the information and incentive gains obtained via the marginal audit.

Leverage is higher under the non-contingent contract than under the efficient standard debt contract, and therefore the marginal resource cost of audits is higher than under the standard debt contract. The marginal benefit from information obtained in additional audits is actually identical under the two locally efficient contracts considered. Under the non-contingent contract, the difference in expected marginal utility across project outcomes is high, suggesting that the gains from insurance should be higher than under the efficient standard debt contract. However, low consumption of low type entrepreneurs also makes Type-I errors particularly costly, preventing significant penalties in auditing contracts, and reducing the benefits obtained by auditing.

The preceding paragraphs explained how small parameter changes—for example a small increase in project risk—can cause the efficient contract to jump from a high leverage, low risk premium, non-contingent contract, to a low leverage, high risk premium and highly contingent standard debt contract.

Within parameter neighbourhoods where defautable debt contracts are efficient, we can use the solutions obtained in appendix A to analyse local perturbations to expected returns, risk, audit costs and audit quality. Consider the following parameterisation: The probability of default is $\underline{\pi} = 1/10$. Conditional upon realisation of the low state, the audit signal returns a high state (a Type-I error) with vanishing probability $\eta(\underline{\theta}) \to 0^+$. The gross opportunity cost of funds $\rho = 21/20$, equivalent to a 5% interest rate. The expected gross return on projects $\mathbb{E}(\theta) = 6/5$. The coefficient of relative risk aversion $\gamma = 1$. Audit costs as a share of the initial assets devoted to the project are $\kappa = 9/80$. In low states the project returns $\underline{\theta} = 33/40$. Subsequently, the high type return is $\bar{\theta} = \frac{1}{\bar{\pi}}[\mathbb{E}(\theta) - \underline{\pi\theta}] = 149/120$. Project risk is $(\bar{\theta} - \underline{\theta}) = 5/12$ and $\zeta = \dfrac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\bar{\pi}(\bar{\theta} - \underline{\theta})} = 37/100$.

By equation 14, leverage (l) is equal to

$$l = 4.$$

By equation 15, the loan interest (coupon) rate is

$$r = 10\%,$$

which can be broken up into the sum of the bank's opportunity cost of funds (5%), compensation for resource costs associated with auditing, (1.5%) and compensation for loss absorbing following verified low reports (3.5%).

We can take derivatives of the log of the leverage ratio to find the semi-elasticities of leverage with respect to expected returns, the intermediary's opportunity cost, risk, audit costs and Type-I errors:

$$\frac{d\log l}{d(\mathbb{E}(\theta))} = 4.23, \qquad \frac{d\log l}{d\rho} = -3.28, \qquad \frac{d\log l}{d(\bar{\theta} - \underline{\theta})} = -3.81,$$

$$\frac{d\log l}{d\kappa} = -0.423 \qquad \text{and} \qquad \frac{d\log l}{d\eta(\underline{\theta})} = -0.189$$

respectively, reported to 3 significant figures.[15]

We can also determine the sensitivity of efficient loan interest rates to underlying parameters:

$$\frac{dr}{d(\mathbb{E}(\theta))} = -0.219, \qquad \frac{dr}{d\rho} = 1.20, \qquad \frac{dr}{d(\bar{\theta} - \underline{\theta})} = 0.197,$$

$$\frac{dr}{d\kappa} = 0.155 \qquad \text{and} \qquad \frac{dr}{d\eta(\underline{\theta})} = -0.0357.$$

Whilst these calculations are specific to our example, they do provide insights into the tradeoffs more generally faced by entrepreneurs.

An increase in expected project returns $\mathbb{E}(\theta)$ increases leverage and, perhaps surprisingly, decreases loan interest rates. The prospect of higher returns encourages entrepreneurs to increase leverage, but they are limited in doing so due to the presence of Type-I errors. A decrease in interest payments in high states r leaves the entrepreneur enough funds in low states to repay loans in full following Type-I errors. In order to satisfy the financial intermediary's participation constraint, providing an expected return of ρ on loans b net of auditing costs, the entrepreneur must absorb more project risk. Low state repayments following verified reports $(\mathcal{Z}(\underline{m}, \underline{\sigma})/b)$ are increased. Decreases in project risk $(\bar{\theta} - \underline{\theta})$ and audit costs (κ) have a similar effect on leverage and interest rates as increases in expected returns $\mathbb{E}(\theta)$.

Following a decrease in the financial intermediary's opportunity cost of funds, loan interest rates fall by an even greater amount. In other words, the spread between loan interest rates and the opportunity cost of funds is increasing in the opportunity cost of funds. First, a decrease in ρ allows the entrepreneur to make lower repayments in all states while meeting the intermediary's participation constraint. Second, when ρ is low the entrepreneur enjoys more of the gains from increased leverage. As in the case of an increase in expected returns $\mathbb{E}(\theta)$, increases in leverage require the entrepreneur to further lower interest repayments such that full repayments are possible even following Type-I errors. To compensate, repayments following verified low reports $(\mathcal{Z}(\underline{m}, \underline{\sigma})/b)$ must increase.

The only variable which moves leverage and loan interest rates in the same direction is audit quality, as measured by the conditional probability of Type-I error $(\eta(\underline{\theta}))$. An increase in the probability of error encourages contracts with greater consumption allocations following disputes. This adjustment is achieved first by decreasing leverage, which at any given interest rate increases the amount of resources remaining following Type-I errors and second by decreasing the loan interest rate r, which further reduces repayments following Type-I errors. This adjustment requires an increase in repayments in low states $(\mathcal{Z}(\underline{m}, \underline{\sigma})/b)$ to compensate the financial intermediary.

[15] For example, an increase in expected returns by 0.01 causes an increase in the optimal leverage ratio by 4.23 percent. Note that rather than reporting semi-elasticities for returns in each state $(\bar{\theta}, \underline{\theta})$, we have reported responses to expected returns $\mathbb{E}(\theta)$ and risk $(\bar{\theta} - \underline{\theta})$, which we find to be more useful for intuition.

7 NUMERICAL EXAMPLE

Our proof of Theorem 2 uses a simplified version of the model to show that standard debt contracts can be maximizers of costly state verification programmes with imperfect audits. In this section, we present a more general, richer version of the model. This richer version of the model allows us to compare the predictions of the model with stylised facts about small business borrowing, and serves to highlight the main tradeoffs determining the choice between debt and equity. The main finding of our numerical exercise is that when audit signals are weakly correlated with true incomes, standard debt contracts emerge as optimal. When audit signals are highly correlated with true incomes, optimal contracts resemble equity.

Audits play two important roles in costly state verification contracts. First, as emphasised by Townsend (1979), audits of low reports allow contract enforcers to punish misreporting agents with high penalties. Second, as emphasised by Border and Sobel (1987), audits allow contract enforcers to reward truth telling agents with low repayments after truthful reports; truth tellers want to be audited. Our numerical example shows both of these mechanisms at work. When audits are high quality, even high reports are audited, and verified truthful reports are rewarded with low repayments. The second mechanism requires interior audit signals: when audit strategies are deterministic, there is no meaningful distinction between repayments following an unaudited report and repayments following an audit consistent with the reported income. In our computed example, the first mechanism is dominant when the audit signal has low quality. Audit strategies are deterministic, with certain audits following low reports. Audit signals are used to punish misreporting agents. The second mechanism emerges as the audit signal quality improves. High reports are audited and truthful reports are rewarded with low repayments relative to unaudited reports.

7.1 SOLUTION STRATEGY

The model is solved numerically for an example with 7 revenue states. For the main numerical simulations, we hold the amount of resources transferred to the entrepreneur b constant to focus on the link between optimal audit strategies and audit quality. This leaves us with $n + n(l + 1)$ choice variables to be determined, where n is the number of revenue states and l is the number of signal states.[16] It follows that in the perfect audits version of our model, there are $7 + 7(7 + 1) = 63$ choice variables. Given the non-convexity of the Programme, the large number of choice variables poses a challenge for combining speed with numerical stability. The algorithm we describe below takes advantage of parallel processing when sampling over candidate audit strategies.

The solution algorithm splits the problem into two loops. The inner loop solves for the

[16] The audit strategy $\mathcal{Q}(m)$ comprises $|M| = |\Theta| = n$ elements; the repayment strategy $\mathcal{Z}(m, \sigma)$ comprises $|M| \times |\Sigma + 1| = n(l + 1)$ states, where the $l + 1$ possible signals include the l signals drawn from the audit technology and the uninformative, null signal $\{\emptyset\}$ drawn when no audit occurs.

optimal repayment allocations $\mathcal{Z}$ conditional upon a candidate solution for the audit strategy $\mathcal{Q}$. This inner loop is solved using a Feasible Directions search algorithm. The outer loop solves for the optimal audit strategy $\mathcal{Q}$ using a combination of global sampling and local perturbation routines. The global sampling routine provides assurance that the solutions found are globally optimal, while the local perturbation routine improves precision.

7.2 PARAMETERS

Gross returns θ are drawn from a discrete uniform distribution unif$\{0.7, 1.5\}$ with 7 draws. The probabilities $\pi(\theta_i)$ are drawn from the binomial distribution $B(6, 0.5)$, $\pi_i = \binom{n}{i-1}/2^n$. The return draw can be considered as a function of smaller positive and negative draws, consistent with standard interpretations of the binomial distribution as the sum of "good" draws from an urn with replacement.[17] The auditor is assumed to observe a subset of the individual draws that determine gross returns. Audits are perfect when the auditor observes all individual draws, and thereby can observe θ with certainty. Audits are imperfect when the auditor observes a strict subset of the individual draws that determine gross returns.[18] Under this structure, the distribution of returns θ conditional upon audit signal σ is $\Delta(\theta_i|\sigma_k) = \binom{n-l}{i-k}/2^{n-l}$ when the number of signals is less than the number of return states $l < n$.

Initial net worth α is a scaling parameter, and is set at 100. Borrowing is fixed at $b = 44.9$, which is taken from the US Federal Reserve Z1 tables.[19] Audit costs are set at $\kappa = 0.02$, which is consistent with empirical estimates of the direct cost of bankruptcy as a share of firm assets between 0.01 and 0.06.[20]

The entrepreneur's coefficient of relative risk aversion $\left(\dfrac{-c\,\mathcal{U}''(c)}{\mathcal{U}'(c)}\right)$ is assumed to be constant, with utility taking the functional form $\mathcal{U}(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$. The coefficient of relative risk aversion is set at $\gamma = 1.5$, which is consistent with the estimates of Herranz et al. (2015). Under this functional form and parameterisation, there is no optimal contract under perfect auditing; allocations approach arbitrarily close to first-best allocations. When $\gamma \geq 1$, $\lim_{c\to 0^+} \mathcal{U}(c) = -\infty$. Given any audit strategy with strictly positive audits in all states ($\mathcal{Q}(m)$ s.t. $\mathcal{Q}(m) > 0\ \forall m$), maximal penalties ensure that all incentive compatibility constraints are relaxed, regardless of allocations and the level of individual audit probabilities. It follows that we can always perturb $\mathcal{Q}$ to reduce audit probabilities in all states, retaining incentive compatibility and expected utility while relaxing the participation constraint of the financial intermediary.

[17]This distribution of returns cannot capture the skewness and kurtosis of entrepreneurial returns in the data as documented by Herranz et al. (2015). Matching this data more closely would require a more complex distribution of states, losing precision over the middle of the distribution, or a large increase in computational time.

[18]For example, let gross returns be affected by both input costs and input quality. It may be the case that the audit signal σ could reveal input costs with certainty, but not input quality.

[19]B.104 Balance Sheet of Nonfinancial Noncorporate Business, FL114104006 Debt as a percentage of net worth, 2016 Q4.

[20]See for example Warner (1977), Weiss (1990) and Altman (1984).

7.3 Results

Result 1 *The model predicts standard debt contracts as globally optimal contracts when the quality of the audit technology is low.*

Under our parameterisation, when the audit technology produces a weak signal standard debt contracts are optimal. When the number of signal states is 1, the audit signal is uninformative, and the optimal contract is non-contingent debt with no auditing. When the number of signal states is 2 or 3, the optimal contract is standard debt, auditing low reports with certainty and not auditing any moderate or high reports. As the number of signal states increases, optimal contracts start to resemble equity contracts and when the audit signal is perfect contracts are able to implement allocations arbitrarily close to the first best allocations.

Subplots (g) and (h) focus on the case where the number of signal states is equal to 2. In this case, the optimal contract takes the form of standard debt, with default following a received income of θ_1. Subplot (g) plots the set of active incentive constraints under the optimal contract, $\{(i, j) \mid \text{ICC}_{i,j} = 0\}$. Between θ_2 and θ_7, the audit probability is zero and repayments are constant. All incentive constraints combining pairs of reports within this range hold with equality. Certain audits of report m_1 ensure that the incentive constraint $\text{ICC}_{2,1}$ is non-binding. Any agent receiving income θ_2 and reporting message m_1 is detected as misreporting with strictly positive penalty and charged a maximal repayment. This is sufficient to ensure that agents receiving θ_2 strictly prefer not to report m_1. However, the audit signal cannot deter with certainty the true income of misreporting agents. The maximal deterrent for an agent receiving θ_2, is not maximal for agents receiving $\theta_3, \theta_4, ..., \theta_7$. Even with certain audits of reported income m_1, the incentive constraint for agents receiving true income θ_7 and considering a report of m_1 ($\text{ICC}_{7,1}$) is binding. This feature of optimal contracts shows how optimal audit probabilities can jump from one to zero following a small increase in reported incomes. This also provides us with an example of the failure of the single crossing condition in costly state verification programmes; ensuring incentive compatibility for adjacent income pairs $\text{ICC}_{k,k-1}$ does not ensure incentive compatibility for all income pairs.

Result 2 *The relationship between interest rate spreads and default probabilities is consistent with empirical estimates.*

When there are $l = 2$ audit signals, the probability of default is 1.56%, the interest rate spread is $\dfrac{z_n(\emptyset)}{b} - \rho = 1.94\%$. When there are $l = 3$ audit signals, the probability of default is 10.94%, the interest rate sread is 11.04%. Note that under our discretization, default probabilities are restricted—agents cannot write a contract with a default probability between 1.56% and 10.94%. For comparison, we can construct approximate annual loan interest rate spread from interest expense and liability data captured by the US Federal Reserve Z1 Tables. From these figures, we can construct approximate annualised loan interest rate spreads of 3.51% and 4.35% where the former estimate constructs the opportunity cost of funds ρ from financial institutions' interest expense and the latter estimate equates the opportunity cost of funds to the London Interbank Offered Rate.[21] The relationship between default probabilities and interest rate spreads predicted by our model seems plausible given the available empirical estimates. Averaging over our two standard debt contracts to match an average interest rate spread of 4.35%, our model predicts an average default probability of 4.04%, within the range of 3.5-4.5% reported as annual average default probabilities of small firms by Herranz et al. (2015).

Result 3 *The shadow value of improved audit technology is high, but less than empirical estimates of the costs of improved auditing.*

In our model, we do not allow entrepreneurs and lenders to choose their audit technology. If the welfare costs of weak audit signals were particularly high, then we might expect agents to make investments in transparency and audit quality in order to attain higher quality audit signals and consequently enjoy greater capacity for risk sharing. Ritter (1987) estimates the direct legal, underwriting and auditing costs of going public for US firms of 14.03% of the proceeds of issuance, which corresponds to 6.30% of initial net worth under our parameterisation. In our model, an entrepreneur constrained by an audit signal revealing 1 of 6 states that determine gross returns ($l = 2$) would be willing to pay 3.35% of initial net wealth in measures increasing the quality of audits such that they correctly identify gross returns with certainty (Subplot (c)). This value is large enough to be economically significant, but smaller than the estimated 6.30% of assets required to be paid for increases in audit quality. Access to an additional margin for improved transparency and audit quality does not break the optimality of standard debt contracts.

Result 4 *The model predicts an optimal leverage ratio that is consistent with empirical estimates; the optimality of standard debt is consistent with optimal leverage choices.*

Subplot (h) provides a further reasonableness check. Recall that for these simulations, we fix the value of borrowing b at 44.9, which is set to equal the average leverage ratio of US noncorporate nonfinancial firms. Under the standard debt contract found for $l = 2$, the optimal optimal borrowing amount b falls between 47.5 and 52.5, close to the exogenously fixed value of 44.9. The optimal audit strategy is constant over these values of borrowing b—access to a margin for increasing and decreasing leverage does not break the optimality of standard debt contracts. Note that by Proposition 4, under this parameterisation with perfect auditing the optimal leverage ratio is infinite.

[21]All values are for 2015, (USD billions). Nonfinancial noncorporate business: Interest Paid, 244.0 (FA116130001), Total Loans, 4746.8 (FL114123005). Financial business: Interest Paid, 1447.1 (FA796130001), Total Liabilities, 88676.2 (FL794194005); 12 month LIBOR 0.79% (St Louis Federal Reserve FRED Database: USD12MD156N).

8 DISCUSSION

Standard debt contracts can be the optimal form of external finance contracts when contract enforcement is uncertain due to noisy audit signals. Supporting truth-telling under stochastic audit strategies requires large penalties. When there is no guarantee that these penalties are fairly applied, these contracts will not be acceptable to risk averse entrepreneurs. The resulting efficient contracts will audit consequent only on low reports, but will likely audit low reports with certainty. As a result, only small penalties are required to ensure truth-telling in equilibrium. In fact, the penalty following a disputed report in an optimal debt contract is typically very close to fully repaying the debt.

Imperfect verification also implies other interesting properties of optimal contracts. For instance, it means that borrowers can only pass a limited amount of risk on to lenders, regardless of contracted audit strategies. Further, even when projects enjoy constant returns to scale and audits are relatively inexpensive, firm size and leverage is endogenously limited by the entrepreneur's risk preference.

Harsanyi (1973) presents an isomorphism between games with mixed strategy Nash equilibria and "disturbed" games with stochastic payoffs and pure strategy Nash equilibria. Our findings resonate with Harsanyi's result; adding randomness to the audit signal encourages pure strategy auditing. However, there is no direct isomorphism between mixed audit strategies under perfect audits and deterministic audit strateges under imperfect audits in our model.

Bolton (1987) shows that in the principal-agent model, noisy audit signals do not overturn the principle of maximum deterrence: the combination of vanishing audit probabilities and vanishing consumption following evidence of shirking is not overturned by the prospect of type-I errors. The main difference between Bolton's setting and our own is that in our setting, it is repayments that are contracted upon, rather than consumption directly. In our model, imposing a maximum penalty on a truth-telling low type does not correspond to a maximum penalty for a misreporting high type. To see why this matters, consider an example where $\mathcal{U}'(0) = +\infty$. As the deterrent approaches its maximum for some report, signal pair, $\mathcal{Z}(m_i, \sigma) \to (\alpha + b)\theta_i$. The direct consumption welfare cost of a marginal further increase in $\mathcal{Z}(m_i, \sigma)$ for the truth-telling low type agent i approaches $-\Delta(\theta_i, \sigma|q_i)\mathcal{U}'(0) = -\infty$. The marginal benefit from improved incentives is $\sum_j \mu_{j,i}\Delta(\theta_j, \sigma|q_i)\mathcal{U}'((\alpha + b)(\theta_j - \theta_i)) < +\infty$, where $\mu_{j,i}$ is the shadow cost of the incentive constraint ICC$_{j,i}$. It is the hidden income $(\alpha + b)(\theta_j - \theta_i)$ that ensures the costs of maximum deterrence exceed the benefits. There is no hidden income in the principal-agent model studied by Bolton. Without hidden income, the marginal benefits and costs of further increases in penalties approach infinity. The incentive gains from maximum deterrence may exceed the welfare costs even as the deterrent approaches the maximum deterrent.

We end with a final observation. The standard debt contracts derived under imperfect monitoring enjoy an additional benefit that we do not formalise. When enforcement is certain, or near certain, incentive compatibility is not sensitive to the risk tolerance of the entrepreneur. That reduces the potential for adverse selection in two forms: First, the preferences of the entrepreneur may be unobservable; and second, the entrepreneur may have access to hidden wealth. The presence of either of these sources of asymmetric information would make it more difficult to employ a stochastic incentive scheme.

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A PROOFS

The proofs of results below are presented in the order they appear in the main text.

Proof of Remark 2 Non-sufficiency: Proposition 7 shows an example of multiplicity of locally optimal contracts, each of which satisfy the first order conditions specified by Programme 1.

Non-necessity: Let $\mathcal{C}$ be a contract with some report m_i such that the audit probability following this report is equal zero, $q_i = 1$. Consider the repayment $z_i(\emptyset)$. This repayment, $z_i(\emptyset)$ occurs with zero probability in expected utility, in the participation constraint and in the incentive compatibility constraint. Perturbing the value of $z_i(\emptyset)$ has no effect on expected utility nor on feasibility. No first order condition exists for the repayment $z_i(\emptyset)$. However, the repayment $z_i(\emptyset)$ does enter with positive probability into the first order condition for q_i; the costs and benefits of a marginal decrease in q_i are dependent on the value of the repayment $z_i(\emptyset)$, which will be paid with positive probability after any such perturbation. Starting from contract $\mathcal{C}$, we can manipulate $z_i(\emptyset)$ in any way, retaining optimality and feasibility but violating the first order condition for q_i reported below:

$$\lambda\Delta(\theta_i)(\alpha+b)\kappa = \sum_\sigma \Delta_{q_i}(\theta_i,\sigma|q_i)\left[\begin{array}{l}\mathcal{U}(\theta_i,m_i,\sigma)+\lambda z_i(\sigma)\\ +\sum_j \mu_{i,j}\mathcal{U}(\theta_i,m_i,\sigma)\\ -\sum_j \mu_{j,i}\dfrac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)}\mathcal{U}(\theta_j,m_i,\sigma)\end{array}\right] + \nu_0(m_i)-\nu_1(m_i).$$

Alternatively, let $\mathcal{C}$ be a contract with the following property: the report-signal pair (m_i,σ) is such that (a) the repayment $\mathcal{Z}(m_i,\sigma)$ enters the second term in the incentive compatibility constraint (7) for an agent receiving revenue shock θ_j and considering misreporting m_i, and (b) is paid with probability zero in truth-telling equilibrium (that is, (a) $\Delta(\theta_j,\sigma|\mathcal{Q}(m_i)) > 0$ and (b) $\Delta(\theta_i,\sigma|\mathcal{Q}(m_i)) = 0$). The first order condition for $\mathcal{Z}(m_i,\sigma)$ cannot be satisfied. If $\mathcal{C}$ is an optimal contract, the repayment $\mathcal{Z}(m_i,\sigma)$ will be sufficiently large such that the incentive constraint is not binding. Any further increases in the repayment $\mathcal{Z}(m_i,\sigma)$ will leave expected utility constant and will not affect feasibility. ■

Proof of Proposition 1 To begin, we split Programme 1 into two distinct optimisation problems. First, the entrepreneur determines the audit strategy $\mathcal{Q}$. Second, the entrepreneur determines the optimal repayment allocations $\mathcal{Z}$, consumption allocations $\mathcal{X}$ and amount of resources initially transferred to the entrepreneur b. The second stage of the problem is reported below as Programme 2:

Programme 2

$$\max_{\mathcal{Z},\mathcal{X},b}\sum_{i,\sigma}\Delta(\theta_i,\sigma|q_i)\mathcal{U}(\theta_i,m_i,\sigma) \tag{16}$$

subject to

$$(\alpha + b)\theta_i - \mathcal{Z}(m_j, \sigma) - \mathcal{X}(\theta_i, m_j, \sigma) = 0 \qquad (\phi_{i,j}(\sigma))$$

$$\sum_{i,\sigma} \Delta(\theta_i, \sigma|q_i) z_i(\sigma) - b\rho - \sum_{i=1}^{n} \Delta(\theta_i) q_i(\alpha + b)\kappa \geq 0 \qquad (\lambda)$$

$$\sum_{\sigma} \Delta(\theta_i, \sigma|q_i)\mathcal{U}(\theta_i, m_i, \sigma) - \sum_{\sigma} \Delta(\theta_i, \sigma|q_j)\mathcal{U}(\theta_i, m_j, \sigma) \geq 0 \qquad (\mu_{i,j})$$

Contemporaneous utility $\mathcal{U}$ is a concave function of consumption allocations $\mathcal{X}$, and therefore Programme 2 is non-convex. Programme 2 and cannot be convexified via re-writing the programme to solve for allocations of $\mathcal{U}$ directly.[22] Before applying the Kuhn-Tucker theorem to generate first order necessary conditions, we must first verify a constraint qualification. We verify the Mangasarian-Fromowitz constraint qualification (Mangasarian and Fromovitz, 1967).

Definition 6 *Given a point $\mathcal{C} \in \mathbb{R}^k$ and the active set $\mathcal{A}(\mathcal{C})$, let $\mathcal{I}(\mathcal{C})$ denote the indices of binding inequality constraints constraints $g_i(\mathcal{C}) = 0$ and equality constraints $h_j(\mathcal{C}) = 0$. The Mangasarian-Fromowitz constraint qualification holds if and only if there exists a vector $d \in \mathbb{R}^k$ such that*

$$\nabla g_i(\mathcal{C})'d > 0 \quad \forall i \in \mathcal{I}(\mathcal{C}), and$$

$$\nabla h_j(\mathcal{C})'d = 0 \quad \forall j \in \mathcal{I}(\mathcal{C}).$$

Lemma 1 *Programme 2 satisfies the Mangasarian-Fromowitz constraint qualification.*

Proof [23] The active set $\mathcal{A}(\mathcal{C})$ of the subproblem consists of the budget constraints, the participation constraint and the active incentive constraints (as the audit probabilities are predetermined, the upper and lower bounds on audit probabilities are not active in the subproblem, Programme 2). We start by reporting the gradients of the equality and inequality constraints:

∇BC:

$$\frac{\partial \mathrm{BC}_{i,j,\sigma}}{\partial b} = -\theta, \qquad \frac{\partial \mathrm{BC}_{i,j,\sigma}}{\partial \mathcal{Z}(m_j, \sigma)} = 1, \qquad \frac{\partial \mathrm{BC}_{i,j,\sigma}}{\partial \mathcal{X}(\theta_i, m_j, \sigma)} = 1.$$

∇PC:

$$\frac{\partial \mathrm{PC}}{\partial b} = -\sum_{\theta} \Delta(\theta) q(\theta)\kappa - \rho, \qquad \frac{\partial \mathrm{PC}}{\partial \mathcal{Z}(m, \sigma)} = \Delta(\theta, \sigma|q(\theta))$$

$\nabla \mathrm{ICC}_{i,j}$:

$$\frac{\partial \mathrm{ICC}_{i,j}}{\partial \mathcal{Z}(m_i, \sigma)} = -\Delta(\theta_i, \sigma|q(m_i))\mathcal{U}'(\theta_i, m_i, \sigma), \qquad \frac{\partial \mathrm{ICC}_{i,j}}{\partial \mathcal{Z}(m_j, \sigma)} = \Delta(\theta_i, \sigma|q(m_j))\mathcal{U}'(\theta_i, m_j, \sigma)$$

[22] This substitution would generate a convex cost function $\mathcal{X}(\cdot) = \mathcal{U}^{-1}(\cdot)$. Recall that the budget constraints in our model are equality constraints, not inequality constraints. The introduction of convex cost function $\mathcal{X}$ would make the budget constraints non-convex.

[23] The proof below does rely on the impermissability of high messages following low revenue states. A more general proof relaxing this assumption is available from the authors by request.

$$\frac{\partial \text{ICC}_{i,j}}{\partial b} = \sum_{\sigma} \Delta(\theta_i, \sigma | q(m_i))\theta_i \mathcal{U}'(\theta_i, m_i, \sigma) - \sum_{\sigma} \Delta(\theta_i, \sigma | q(m_j))\theta_i \mathcal{U}'(\theta_i, m_j, \sigma)$$

We wish to construct a vector d, specified by the definition 6. Now, let $\mathcal{D} : \mathcal{C} \to \mathbb{R}$ be a mapping from the vector of contract terms in $\mathcal{C}$ to the reals. The elements of vector d are represented by the images of function $\mathcal{D}$, such that without loss of generality, $d_i = \mathcal{D}(\mathcal{Z}(m, \sigma))$ iff $\mathcal{C}_i = \mathcal{Z}(m, \sigma)$. We search for some function $\mathcal{D}$ such that

$$\sum_{c \in \mathcal{C}} \frac{\partial g_i}{\partial c} \mathcal{D}(c) > 0, \qquad \textit{and} \qquad \sum_{c \in \mathcal{C}} \frac{\partial h_j}{\partial c} \mathcal{D}(c) = 0.$$

We construct $\mathcal{D}$ with three steps. First, we set $\mathcal{D}(b)$. Second, we set $\mathcal{D}(\mathcal{Z})$ and finally we set $\mathcal{D}(\mathcal{X})$

First, set

$$\mathcal{D}(b) = 0. \tag{17}$$

Second, we set the values associated with repayment allocations $\mathcal{D}(\mathcal{Z}(m_i, \sigma))$:

Choose some signal σ'' such that $\Delta(\theta_j, \sigma'' | q_j) > 0$. Set $\mathcal{D}(\mathcal{Z}(m_j, \sigma'')) > 0$. We now have

$$\sum_{c \in \mathcal{C}} \frac{\partial \text{PC}}{\partial c} \mathcal{D}(c) > 0.$$

This inequality will be retained by ensuring non-negativity of all $\mathcal{D}(\mathcal{Z}(m, \sigma))$.

Now, for $i \in (2, 3, ..., n)$ and $j \in \{1, 2, ..., i\}$, choose some signal $\sigma'_{i,j}$ such that $\Delta(\theta_i, \sigma'_{i,j} | q_j) > 0$. Set $\mathcal{D}(\mathcal{Z}(m_j, \sigma'_{i,j}))$ sufficiently high such that

$$\sum_{\sigma} \left[\frac{\partial \text{ICC}_{i,j}}{\partial \mathcal{Z}(m_i, \sigma)} \mathcal{D}(\mathcal{Z}(m_i, \sigma)) + \frac{\partial \text{ICC}_{i,j}}{\partial \mathcal{Z}(m_j, \sigma)} \mathcal{D}(\mathcal{Z}(m_j, \sigma)) \right] > 0. \tag{18}$$

Third and finally, we need to satisfy the Mangasarian-Fromowitz conditions for the budget constraints. These constraints are equality constraints, therefore we set

$$\mathcal{D}(\mathcal{X}(\theta_i, m_j, \sigma)) = -\mathcal{D}(\mathcal{Z}(m_j, \sigma)) \tag{19}$$

This ensures

$$\sum_{c, \theta_i, m_j, \sigma} \frac{\partial \text{BC}_{i,j,\sigma}}{\partial c} \mathcal{D}(c) = 0,$$

Completing the proof. ■

Lemma 1 allows us to apply the Kuhn-Tucker theorem to derive necessary conditions for Programme 2 using a first-order approach.[24]

[24] See, for example Bertsekas (1995).

The first order necessary conditions Programme 2 are

$$\begin{aligned} z_i(\sigma): \quad & 0 = -\Delta(\theta_i, \sigma|q_i)\mathcal{U}'(\theta_i, m_i, \sigma) + \lambda\Delta(\theta_i, \sigma|q_i) \\ & \quad - \sum_j \mu_{i,j}\Delta(\theta_i, \sigma|q_i)\mathcal{U}'(\theta_i, m_i, \sigma) + \sum_j \mu_{j,i}\Delta(\theta_j, \sigma|q_i)\mathcal{U}'(\theta_j, m_i, \sigma) \qquad (20) \\ b: \quad & 0 = \sum_{i,\sigma} \Delta(\theta_i, \sigma|q_i)\theta_i\mathcal{U}'(\theta_i, m_i, \sigma) - \lambda\left[\rho + \sum_{i=1}^n \Delta(\theta_i)q_i\kappa\right] \\ & \quad + \sum_{i,j} \mu_{i,j}\theta_i\left[\sum_\sigma \Delta(\theta_i, \sigma|q_i)\mathcal{U}'(\theta_i, m_i, \sigma) - \sum_\sigma \Delta(\theta_i, \sigma|q_j)\mathcal{U}'(\theta_i, m_j, \sigma)\right] \qquad (21) \end{aligned}$$

By Bayes' Theorem, we can re-write $\Delta(\theta_i, \sigma|q_j) = \Delta(\sigma(q_j)|\theta_i)\Delta(\theta_i) = \Delta(\theta_i|\sigma(q_j))\Delta(\sigma(q_j))$. It follows that we have

$$\frac{\Delta(\theta_i, \sigma|q_j)}{\Delta(\theta_j, \sigma|q_j)} = \frac{\Delta(\theta_i|\sigma)}{\Delta(\theta_j|\sigma)}.$$

We can now divide Equation 20 by $\Delta(\theta_i, \sigma|q_i)$ and rearrange to obtain Equation 9.

$$\frac{\mathcal{U}'(\theta_i, m_i, \sigma)}{\lambda} = \frac{1}{1+\sum_j \mu_{i,j}}\left[1 + \sum_j \mu_{j,i}\frac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)}\frac{\mathcal{U}'(\theta_j, m_i, \sigma)}{\lambda}\right] \qquad (9)$$

Use Equation 20 to eliminate the terms

$$\sum_{i,\sigma} \Delta(\theta_i, \sigma|q_i)\theta_i\mathcal{U}'(\theta_i, m_i, \sigma) + \sum_{i,j} \mu_{i,j}\theta_i \sum_\sigma \Delta(\theta_i, \sigma|q_i)\mathcal{U}'(\theta_i, m_i, \sigma)$$

from Equation 21:

$$\begin{aligned} 0 = & \sum_i \theta_i \sum_\sigma \lambda\Delta(\theta_i, \sigma|q_i) - \lambda\left[\rho + \sum_{i=1}^n \Delta(\theta_i)q_i\kappa\right] \\ & + \sum_i \theta_i \sum_\sigma \sum_j \mu_{j,i}\Delta(\theta_j, \sigma|q_i)\mathcal{U}'(\theta_j, m_i, \sigma) \\ & - \sum_i \theta_i \sum_\sigma \sum_j \mu_{i,j}\Delta(\theta_i, \sigma|q_j)\mathcal{U}'(\theta_i, m_j, \sigma) \end{aligned}$$

Re-labelling the summands in the third term (swapping the i's and j's) allows us to simplify

$$\begin{aligned} 0 = & \sum_i \theta_i \sum_\sigma \lambda\Delta(\theta_i, \sigma|q_i) - \lambda\left[\rho + \sum_{i=1}^n \Delta(\theta_i)q_i\kappa\right] \\ & + \sum_{i,j} \sum_\sigma \Delta(\theta_i, \sigma|q_j)\mu_{i,j}(\theta_j - \theta_i)\mathcal{U}'(\theta_i, m_j, \sigma). \end{aligned}$$

The first term is just the product of the shadow cost of the participation constraint λ and expected returns.[25] Collecting terms in λ we obtain Equation 10:

$$\lambda\left[\sum_i \Delta(\theta_i)(\theta_i - q_i\kappa) - \rho\right] = \sum_{i,j}\sum_{\sigma}\Delta(\theta_i,\sigma|q_j)\mu_{i,j}(\theta_i-\theta_j)\mathcal{U}'(\theta_i,m_j,\sigma).$$

■

Proof of Corollary 1

Part (a): Part (a) follows from inspection of Equation 9. If $\text{ICC}_{j,i} \notin \mathcal{A}(\mathcal{C})\ \forall j$ then Equation 9 becomes

$$\mathcal{U}'(\theta_i,m_i,\sigma) = \frac{\lambda}{1+\sum_j \mu_{i,j}},$$

which is constant for all σ.

Part (b): Consider revenue shock θ_i such that there exists some shock θ_j with the property that $\text{ICC}_{j,i} \in \mathcal{A}(\mathcal{C})$. Start with the First Order Necessary condition for repayment allocations $\mathcal{Z}(m_i,\sigma)$, Equation 9, which is repeated below for convenience:

$$0 = -\left(1+\sum_k \mu_{i,k}\right)\mathcal{U}'(\theta_i,m_i,\sigma) + \lambda + \sum_j \mu_{j,i}\frac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)}\mathcal{U}'(\theta_j,m_i,\sigma)$$

We're interesting in seeing how the repayment allocation $\mathcal{Z}(m_i,\sigma)$ must respond to fluctuations in the marginal likelihood ratio, $\Gamma_i^j(\sigma) := \dfrac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)}$. Holding i constant, the remaining terms in μ,λ are held constant as we vary the repayment signal σ. Take the total derivative with respect to Γ and the repayment allocation $\mathcal{Z}(m_i,\sigma)$:

$$\begin{aligned}0 = &\left(1+\sum_k \mu_{i,k}\right)\mathcal{U}''(\theta_i,m_i,\sigma)d\mathcal{Z}(m_i,\sigma) - \sum_j \mu_{j,i}\Gamma_i^j(\sigma)\mathcal{U}''(\theta_j,m_i,\sigma)d\mathcal{Z}(m_i,\sigma)\\ &+\sum_j \mu_{j,i}\mathcal{U}'(\theta_j,m_i,\sigma)d\Gamma_i^j(\sigma)\end{aligned}$$

Divide through by $(1+\sum_k \mu_{i,k})\mathcal{U}'(\theta_i,m_i,\sigma)\left(= \lambda + \sum_j \mu_{j,i}\dfrac{\Delta(\theta_j|\sigma)}{\Delta(\theta_i|\sigma)}\mathcal{U}'(\theta_j,m_i,\sigma) \text{ by Equation 9}\right)$ and substitute $A(x) = \dfrac{-\mathcal{U}''(x)}{\mathcal{U}'(x)}$

$$\left[A(\theta_i,m_i,\sigma) - \frac{\sum_j \mu_{j,i}\Gamma_i^j(\sigma)}{\lambda+\sum_j \mu_{j,i}\Gamma_i^j(\sigma)}A(\theta_j,m_i,\sigma)\right]d\mathcal{Z}(m_i,\sigma) = \frac{\sum_j \mu_{j,i}}{1+\sum_k \mu_{i,k}}\frac{\mathcal{U}'(\theta_j,m_i,\sigma)}{\mathcal{U}'(\theta_i,m_i,\sigma)}d\Gamma_i^j(\sigma)$$

By the assumption of non-increasing absolute risk aversion, $A(\theta_j,m_i,\sigma) \le A(\theta_i,m_i,\sigma)$. The The multipliers $\mu\lambda$ and the likelihood ratios Γ are all non-negative, therefore the difference

[25] $\left(\sum_i \theta_i \sum_\sigma \Delta(\theta_i,\sigma|q_i) = \sum_{i,\sigma}\Delta(\theta_i,\sigma|q_i)\theta = \sum_i \Delta(\theta_i)\theta\right)$ by the law of total probability.

$A(\theta_i, m_i, \sigma) - \dfrac{\sum_j \mu_{j,i}\Gamma_i^j(\sigma)}{\lambda + \sum_j \mu_{j,i}\Gamma_i^j(\sigma)} A(\theta_j, m_i, \sigma)$ and the ratio $\dfrac{\sum_j \mu_{j,i}}{1 + \sum_k \mu_{i,k}} \dfrac{\mathcal{U}'(\theta_j, m_i, \sigma)}{\mathcal{U}'(\theta_i, m_i, \sigma)}$ are strictly positive. It follows that $\dfrac{d\Gamma_i^j(\sigma)}{d\mathcal{Z}(m_i, \sigma)} > 0$, completing the proof.

■

Proof of proposition 2

a. Let the financial intermediary's participation constraint be non-binding. Let σ be a signal with the property that $\Delta(\theta_n, \sigma | q_n) > 0$. We can reduce the repayment $\mathcal{Z}(m_n, \sigma)$, which will tighten the currently slack participation constraint, increase expected utility, and relax the incentive constraints ICC$_{n,i}$ for $i \in \{1, 2, 3, ..., n-1\}$.

b. The highest possible report m_n only appears on the left hand side of incentive constraints. By Corollary 1, the optimal repayment allocation $\mathcal{Z}(m_n, \sigma)$ is not contingent on the audit signal σ. It follows that we can remove all audits of reports m_n without affecting expected utility or incentive constraints, but relaxing the participation constraint for any positive audit cost.

c. Let $\mathcal{C}$ be a contract with no binding incentive constraints.

   Case i: If all audit probabilities are equal to zero, $\mathcal{Q}(m) = 0, \ \forall m$, then we can verify the proposition directly from the first order condition for $\mathcal{Z}$. When no audits are undertaken, incentive compatiblity requires $\mathcal{X}(\theta_i, m_i, \emptyset) = \mathcal{X}(\theta_j, m_j, \emptyset) + (\alpha + b)(\theta_i - \theta_j)$ by equation 7. If no incentive constraints are binding, then the first order condition for $\mathcal{Z}$ states that $\mathcal{U}'(\theta_i, m_i, \emptyset) = \lambda = \mathcal{U}'(\theta_j, m_j, \emptyset)$, a contradiction.

   Case ii: If there exists a positive audit probability for some report $\mathcal{Q}(m_i) > 0$, then we can reduce this audit probability, replacing the foregone audits with a lottery replicating the distribution of the true audit technology conditional upon truth-telling. For details, see the proof of Theorem 1.

d. If repayment allocations are not contingent on audit signals, then we can apply the same argument from part (c)(i).

   If repayment allocations are contingent on audit signals, then there exists i, σ, σ' such that $\mathcal{X}(\theta_i, m_i, \sigma) \neq \mathcal{X}(\theta_i, m_i, \sigma')$. It follows that $\mathcal{U}'(\theta_i, m_i, \sigma) \neq \mathcal{U}'(\theta_i, m_i, \sigma')$. This is consistent with the first order condition for repayment allocations (9) only if there exists some j such that $\mu_{j,i} > 0$.

■

Proof of Theorem 1 Assume audits are perfect with strictly positive cost ($\Delta(\theta|\sigma) \in \{0, 1\}$ and $\kappa > 0$). Let $\mathcal{C}$ be a truth-telling contract with the feature that there is some report m_i which is followed by a certain audit, $q_i = 1$. Assume that under contract $\mathcal{C}$, any truth-telling agent enjoys some strictly positive probability of avoiding immiseration: $\forall\, i, \ \sum_\sigma \Delta(\theta_i, \sigma | q_i) \mathcal{U}(\theta_i, m_i, \sigma) > \underline{\mathcal{U}}$.

Perturbation 1 Consider the set of repayment allocations contingent on reported income m_i that occur with zero probability under truth-telling. Set all of the repayment allocations in this set equal to their corresponding maximal repayments:

$$\text{if } \Delta(\theta_i|\sigma) = 0 \text{ and } \Delta(\theta_j|\sigma) = 1, \text{ then } \mathcal{Z}'(m_i, \sigma) = (\alpha + b)\theta_j.$$

Following perturbation 1, any agent who misreports θ_i is immiserated with certainty. By assumption, truth-telling agents are not immiserated with positive probability. It follows that the incentive constraints $\text{ICC}_{j,i}$ are non-binding for all j.

Perturbation 2 Following the reported message m_i, employ the audit technology with probability $(1 - s)$. With probability s, draw a signal σ_s from the distribution $\sigma_s \sim \Sigma(\theta_i)$, that is, let $\Delta(\sigma_s|\theta_i) = \Delta(\sigma|\theta_i)$ for all values of σ_s, σ. Retain the same repayment schedule, but now condition repayments on either σ or σ_s, whichever is drawn, $\mathcal{Z}''(m_i, \sigma \cup \sigma_s) = \mathcal{Z}'(m_i, \sigma)$.

The new signal has no resource cost. However, the new strategy saves audit costs worth $\Delta(\theta_i)s(\alpha + b)\kappa$. This saving relaxes the participation constraint of the financial intermediary. Agents who truthfully report message m_i experience no change in their conditional distribution of repayments and therefore no change in expected utility. This also implies that for agents receiving return θ_i, their opportunity cost of misreporting remains constant, the perturbation does not violate any incentive constraint of the form $\text{ICC}_{i,j}$. Following perturbation 1, incentive constraints of the form $\text{ICC}_{j,i}$ are slack before Perturbation 2, and remain slack for sufficiently small s.

Taken together, Perturbations 1 and 2 relax the financial intermediaries' participation constraint while maintaining incentive compatibility and expected utility. We can continue to apply Perturbations 1 and 2 to each report that motivates certain auditing under the initial contract $\mathcal{C}$ until all audit probabilities are strictly below one. ■

Proof of Proposition 3 We prove the proposition by solving for the contract that implements full insurance with the lowest possible total audit cost. When audit costs are zero, the full insurance, first best, allocation is feasible and optimal. However, when audit costs are zero, excessive audits do not incur welfare costs. When audit costs are strictly positive but approaching zero, optimal repayment allocations will approach the first best allocations, but audit strategies will ration audits to reduce resource costs.

Below is the incentive constraint for an agent receiving true return i and considering whether to report message j (Equation 7):

$$\sum_{\sigma} \Delta(\theta_i, \sigma|q_i)\mathcal{U}(\theta_i, m_i, \sigma) \geq \sum_{\sigma} \Delta(\theta_i, \sigma|q_j)\mathcal{U}(\theta_i, m_j, \sigma)$$

Denote the full insurance consumption bundle by x^*. Under the full insurance allocation, agent i receives allocation x^* under truth telling with certainty, so we can replace the left hand side with $\mathcal{U}(x^*)$. Following a false report of message m_j, agent i would receive consumption allocation $x^* + (\alpha + b)(\theta_i - \theta_j)$ if they are not audited. This represents the full insurance allocation x^* received by any truth telling agent j, plus the difference in revenue between outturns i and j. If audited, an agent i who misreports m_j would have their true return revealed with certainty under perfect audits, and would be charged a maximal penalty. This agent would receive utility allocation $\mathcal{U}_{\min}$, equal to the lower support of the range of the utility function $\mathcal{U}$ if a lower support exists. If a lower support does not exist, $\mathcal{U}_{\min}$ can be set arbitratily low. The incentive constraint can now be re-written as follows:

$$\mathcal{U}(x^*) = (1 - q_j)\,\mathcal{U}(x^* + (\alpha + b)(\theta_i - \theta_j)) + q_j\,\mathcal{U}_{\min}.$$

We can solve explicitly for the minimal audit probability necessary to attain incentive compatibility:

$$q_j = \frac{\mathcal{U}(x^* + (\alpha + b)(\theta_i - \theta_j)) - \mathcal{U}(x^*)}{\mathcal{U}(x^* + (\alpha + b)(\theta_i - \theta_j)) - \mathcal{U}_{\min}}.$$

This condition gives us the limiting optimal audit probability as audit costs approach zero. If the range of $\mathcal{U}$ is unbounded below, incentive compatibility can be attained with q_j arbitrarily close to 0. If the range of $\mathcal{U}$ is bounded below, we can attain incentive compatiblity with strictly interior q_j. ■

Proof of Proposition 4 To complete the proof, we use Proposition 1 to argue directly from the first order conditions (9) and (10). Set all audit probabilities equal to one, $\mathcal{Q}(\theta) = 1$. Under perfect audits, this leaves $\Delta(\theta, \sigma) \in \{0, 1\}\ \forall(\theta, \sigma)$. It follows that the first order condition for repayments $\mathcal{Z}$ expressed as Equation 9 can only be satisfied when $\mu_{i,j} = 0\ \forall j, i$. If we substitute this result into Equation 10, the right hand side is equal to zero. It follows that when $\kappa < \mathbb{E}(\theta) - \rho$, the first order condition for the borrowing cannot be satisfied: the marginal value of additional borrowing always exceeds the marginal cost. To show that entrepreneur consumption is also infinite, substitute this result into Constraints 1 and 3. ■

Proof of Theorem 2 Let borrowing be taken as given $b = \hat{b}$. Let the utility function exhibit non-increasing absolute risk aversion over consumption. When Type-I audit errors occur with positive probability ($\eta(\underline{\theta}) > 0$), there exists some strictly positive audit cost $\hat{\kappa}$ such that for all $\kappa < \hat{\kappa}$, standard debt contracts ($q(\underline{m}) = 1$) are efficient.

We consider an arbitrary efficient contract with interior audit probability, $q(\underline{m}) \in (0, 1)$. We show that if audit costs were sufficiently low, the initial contract could be strictly improved by an increase in audit probability $q(\underline{m})$. As we are only considering allocations consistent with truth-telling, we will drop the report variable from the consumption allocation $\mathcal{X}(\underline{\theta}, \sigma) := \mathcal{X}(\underline{\theta}, \underline{m}, \sigma)$.

Here it is important that audits are imperfect. By Proposition 2 part (d), we know that for any audit probability q_1, the incentive constraint is binding. Corollary 1 part (b) states that when the incentive constraint is binding, the optimal repayment allocation following low reports is contingent on the signal revealed by audits, regardless of the audit strategy.

The contingency of optimal repayments on audit signals ensures that the marginal value of the information contained in the audit signal is strictly positive, regardless of the audit probability q_1. It follows that for κ sufficiently low, the optimal audit probability following low reports is one, $q_1^* = 1$. ■

Proof of Proposition 5 From the budget constraints 1, we have $\mathcal{X}(\theta_j, m_i, \sigma) = \mathcal{X}(\theta_i, m_i, \sigma) + (\alpha + b)(\theta_j - \theta_i)$. By assumption $\Delta(\theta_i, \sigma | q_i) > 0 \ \forall \ \theta_i, \sigma$. If consumption is non-negative for all truth-telling states, $\mathcal{X}(\theta_i, m_i, \sigma) \geq 0 \ \forall \ \theta_i, \sigma$, then it follows that the expectation on the right hand side of the incentive compatibility constraint ICC$_{j,i}$ is bounded below as follows: $\sum_\sigma(\theta_j, \sigma | q_i)\mathcal{U}(\theta_j, m_i, \sigma) \geq \mathcal{U}((\alpha + b)(\theta_j - \theta_i))$. Incentive compatiblity requires

$$\sum_\sigma(\theta_j, \sigma|q_j)\mathcal{U}(\theta_j, m_j, \sigma) \geq \sum_\sigma(\theta_j, \sigma|q_i)\mathcal{U}(\theta_j, m_i, \sigma) \geq \mathcal{U}((\alpha + b)(\theta_j - \theta_i)),$$

which implies

$$\sum_\sigma(\theta_j, \sigma|q_j)\mathcal{X}(\theta_j, m_j, \sigma) \geq (\alpha + b)(\theta_j - \theta_i)$$

by Jensen's inequality. Using the budget constraint 1, we convert this inequality into an upper bound on repayments:

$$\sum_\sigma(\theta_j, \sigma|q_j)[(\alpha + b)\theta_j - \mathcal{Z}(m_j, \sigma)] \geq (\alpha + b)(\theta_j - \theta_i)$$

$$\sum_\sigma(\theta_j, \sigma|q_j)\mathcal{Z}(m_j, \sigma) \leq (\alpha + b)\theta_i$$

Now, substituting this into the participation constraint (5),

$$\sum_{i,\sigma}(\theta_i, \sigma|q_i)\mathcal{Z}(m_i, \sigma) \leq (\alpha + b)\theta_1,$$

$$b\rho + \sum_{i=1}^{n} \Delta(\theta_i)q_i(\alpha + b)\kappa \leq (\alpha + b)\theta_1,$$

which can be rearranged to complete the proof.

■

Proof of Proposition 6 Here we solve for efficient allocations and borrowing when efficient audit strategies are deterministic, ie. when $q = 0$ *or* 1. Here, we assume CRRA utility, $\mathcal{U}(x) = x^{1-\gamma}/(1-\gamma)$. In the text, we refer to the more tractable case of logarithmic utility which can be found by setting $\gamma = 1$ in any of the solutions contained in this section.

NON-CONTINGENT CONTRACTS $(q = 0)$

When audits are not used $(q = 0)$, efficient contracts are non-contingent. Repayments are independent of the entrepreneur's report. No audit signals are obtained. This certainty of repayment makes the incentive compatibility constraint linear in the choice variables, enabling us to solve the entrepreneur's problem with a Lagrangian:

$$\begin{aligned}\mathcal{L}_0 = \bar{\pi}\,\mathcal{U}(\mathcal{X}(\bar{\theta},\bar{m},\emptyset)) + \underline{\pi}\,\mathcal{U}(\mathcal{X}(\underline{\theta},\underline{m},\emptyset))\\ + \lambda[(\alpha+b)(\mathbb{E}(\theta)-\rho)+\alpha\rho-\bar{\pi}\mathcal{X}(\bar{\theta},\bar{m},\emptyset)-\underline{\pi}\mathcal{X}(\underline{\theta},\underline{m},\emptyset)]\\ + \mu\left[\mathcal{U}(\mathcal{X}(\bar{\theta},\bar{m},\emptyset))-\mathcal{U}[\mathcal{X}(\underline{\theta},\underline{m},\emptyset)+(\alpha+b)(\bar{\theta}-\underline{\theta})]\right].\end{aligned} \tag{22}$$

The first order conditions are

$$\begin{array}{llll}\mathcal{X}(\bar{\theta},\bar{m},\emptyset) & : & 0 & = \bar{\pi}\mathcal{U}'(\mathcal{X}(\bar{\theta},\bar{m},\emptyset)) - \bar{\pi}\lambda + \mu\,\mathcal{U}'(\mathcal{X}(\bar{\theta},\bar{m},\emptyset))\\ \mathcal{X}(\underline{\theta},\underline{m},\emptyset) & : & 0 & = \underline{\pi}\mathcal{U}'(\mathcal{X}(\underline{\theta},\underline{m},\emptyset)) - \underline{\pi}\lambda - \mu\,\mathcal{U}'[\mathcal{X}(\underline{\theta},\underline{m},\emptyset)+(\alpha+b)(\bar{\theta}-\underline{\theta})]\\ b & : & 0 & = \lambda(\mathbb{E}(\theta)-\rho) - \mu(\bar{\theta}-\underline{\theta})\mathcal{U}'[\mathcal{X}(\underline{\theta},\underline{m},\emptyset)+(\alpha+b)(\bar{\theta}-\underline{\theta})].\end{array}$$

Substituting the incentive compatibility constraint and utility function into the first order conditions yields

$$\frac{\mathcal{U}'(\mathcal{X}(\underline{\theta},\underline{m},\emptyset))}{\mathcal{U}'(\mathcal{X}(\bar{\theta},\bar{m},\emptyset))} = \frac{\bar{\pi}}{\underline{\pi}}\left(\frac{\bar{\theta}-\rho}{\rho-\underline{\theta}}\right). \tag{23}$$

The left hand side of equation 23 is the entrepreneur's consumption marginal rate of substitution across high and low states. The right hand side is the marginal rate of transformation between consumption in high and low states. An increase in leverage increases consumption in high states and reduces consumption in low states. Equation 23 along with the incentive compatibility constraint can be substituted into the participation constraint to solve first for $\mathcal{X}(\bar{m},\emptyset,\bar{\theta})$ and the remaining choice variables:

$$\begin{aligned}\mathcal{X}(\bar{\theta},\bar{m},\emptyset) &= \frac{\alpha\rho\,(\bar{\theta}-\underline{\theta})}{\left(\frac{\underline{\pi}(\rho-\underline{\theta})}{\bar{\pi}(\bar{\theta}-\rho)}\right)^{1/\gamma}(\bar{\theta}-\rho)+(\rho-\underline{\theta})}\\ \mathcal{X}(\underline{\theta},\underline{m},\emptyset) &= \frac{\alpha\rho\,(\bar{\theta}-\underline{\theta})}{(\bar{\theta}-\rho)+(\rho-\underline{\theta})\left(\frac{\bar{\pi}(\bar{\theta}-\rho)}{\underline{\pi}(\rho-\underline{\theta})}\right)^{1/\gamma}}\\ b &= \alpha\rho\left[\frac{1-\left(\frac{\underline{\pi}(\rho-\underline{\theta})}{\bar{\pi}(\bar{\theta}-\rho)}\right)^{1/\gamma}}{\left(\frac{\underline{\pi}(\rho-\underline{\theta})}{\bar{\pi}(\bar{\theta}-\rho)}\right)^{1/\gamma}(\bar{\theta}-\rho)+(\rho-\underline{\theta})}\right]-\alpha\\ \lambda &= \left[\frac{\underline{\pi}^{1/\gamma}(\bar{\theta}-\rho)^{\frac{\gamma-1}{\gamma}}+\bar{\pi}^{1/\gamma}(\rho-\underline{\theta})^{\frac{\gamma-1}{\gamma}}}{\alpha\rho\,(\bar{\theta}-\underline{\theta})^{\frac{\gamma-1}{\gamma}}}\right]^{\gamma}\\ \mu &= \frac{\bar{\pi}(\mathbb{E}(\theta)-\rho)}{\rho-\underline{\theta}}.\end{aligned} \tag{24}$$

STANDARD DEBT CONTRACTS $(q=1)$ WITH NO TYPE-II ERRORS $(\eta(\bar{\theta})=0)$

When audits occur with certainty following low type reports and the audit signal correctly identifies high type entrepreneurs with certainty, then as in the private information case the incentive compatibility constraint becomes linear. Our problem can be expressed by the following Lagrangian

$$\begin{aligned}\mathcal{L}_1 = {} & \bar{\pi}\, \mathcal{U}(\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset)) + \underline{\pi}(1-\eta(\underline{\theta}))\, \mathcal{U}(\mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma})) + \underline{\pi}\eta(\underline{\theta})\, \mathcal{U}(\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma})) \\ & + \lambda \left[\begin{array}{l} (\alpha+b)(\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa) + \alpha\rho \\ \quad -\bar{\pi}\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset) - \underline{\pi}(1-\eta(\underline{\theta}))\mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma}) - \underline{\pi}\eta(\underline{\theta})\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) \end{array} \right] \\ & + \mu \left[\mathcal{U}(\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset)) - \mathcal{U}[\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) + (\alpha+b)(\bar{\theta} - \underline{\theta})] \right] \end{aligned} \tag{25}$$

The first order conditions are

$$\begin{array}{llcl} \mathcal{X}(\bar{\theta}, \bar{m}, \emptyset) & : \; 0 & = & \bar{\pi}\mathcal{U}'(\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset)) - \bar{\pi}\lambda + \mu\,\mathcal{U}'(\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset)) \\ \mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma}) & : \; 0 & = & \underline{\pi}(1-\eta(\underline{\theta}))\mathcal{U}'(\mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma})) - \underline{\pi}(1-\eta(\underline{\theta}))\lambda \\ \mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) & : \; 0 & = & \underline{\pi}\eta(\underline{\theta})\mathcal{U}'(\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma})) - \underline{\pi}\eta(\underline{\theta})\lambda - \mu\,\mathcal{U}'[\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) + (\alpha+b)(\bar{\theta} - \underline{\theta})] \\ b & : \; 0 & = & \lambda(\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa) - \mu(\bar{\theta} - \underline{\theta})\mathcal{U}'[\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) + (\alpha+b)(\bar{\theta} - \underline{\theta})]. \end{array}$$

Substituting the incentive compatibility constraint into the first order conditions yields the following conditions relating the entrepreneur's consumption marginal rates of substitution across states to their respective marginal rates of transformation of consumption across states:

$$\begin{aligned} & \mathcal{U}'(\mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma})) = \lambda \\ & \frac{\mathcal{U}'(\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset))}{\mathcal{U}'(\mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma}))} = 1 - \frac{1}{\bar{\pi}}\left[\frac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\bar{\theta} - \underline{\theta}}\right] \\ & \frac{\mathcal{U}'(\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}))}{\mathcal{U}'(\mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma}))} = 1 + \frac{1}{\underline{\pi}\eta(\underline{\theta})}\left[\frac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\bar{\theta} - \underline{\theta}}\right]. \end{aligned}$$

We can combine the above conditions with the intermediary's participation constraint and the utility function and solve for consumption allocations:

$$\begin{aligned} \mathcal{X}(\bar{\theta}, \bar{m}, \emptyset) &= \alpha\rho\chi\left(\frac{1}{1-\zeta}\right)^{1/\gamma} \\ \mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma}) &= \alpha\rho\chi \\ \mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) &= \alpha\rho\chi\left(\frac{\underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right)^{1/\gamma} \\ b &= \frac{\alpha\rho\chi}{\bar{\theta} - \underline{\theta}}\left(\frac{\zeta}{1-\zeta}\right)^{1/\gamma}\left(\frac{\bar{\pi} + \underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right)^{1/\gamma} - \alpha \\ \lambda &= (\alpha\rho\chi)^{-\gamma} \\ \mu &= \frac{\bar{\pi}\zeta}{1-\zeta}. \end{aligned} \tag{26}$$

where

$$\chi = \frac{1}{\bar{\pi}\,(1-\zeta)^{\frac{\gamma-1}{\gamma}} + \underline{\pi}\eta(\underline{\theta})\left(\frac{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}{\underline{\pi}\eta(\underline{\theta})}\right)^{\frac{\gamma-1}{\gamma}} + \underline{\pi}(1-\eta(\underline{\theta}))}, \qquad \text{and}$$

$$\zeta = \frac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\bar{\pi}(\bar{\theta} - \underline{\theta})}.$$

■

Proof of Proposition 7

Consider the case where the entrepreneur enjoys consumption with log utility, type–II errors occur with zero probability ($\eta(\bar{\theta}) = 0$) and Type-I errors occur with positive but very low probability, $\eta(\underline{\theta}) \to 0^+$. By proposition 6, consumption following overturned low type reports also tends toward zero ($\lim_{\eta(\underline{\theta})\to 0^+} \mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma}) = 0$). By l'Hôpital's rule, the contribution to ex ante expected utility of the entrepreneur from consumption following overturned reports also tends toward zero, ($\lim_{\eta(\underline{\theta})\to 0^+} \underline{\pi}\eta(\underline{\theta})\mathcal{U}(\mathcal{X}(\underline{\theta}, \underline{m}, \bar{\sigma})) = 0$). When Type-I errors are extremely rare, errors have little effect on the ex ante welfare of entrepreneurs but still limit repayments drawn from high type entrepreneurs.

For tractability, this section restricts analysis in accordance with the following assumptions:

Assumption 3 *Type-I errors will occur with very low (positive) probability, $\eta(\underline{\theta}) \to 0^+$.*

Assumption 4 *Utility will be logarithmic over consumption, $\mathcal{U}(\mathcal{X}(\theta, m, \sigma)) = \log \mathcal{X}(\theta, m, \sigma)$.*

Assumption 5 *The two income states will occur with equal probability, $\bar{\pi} = \underline{\pi} = \frac{1}{2}$.*

Let y_1 (y_0) be the efficient value of choice variable y in the always audit (private information) contract. Substituting $\mathcal{U}(x) = \log x$ and $\underline{\pi} = \bar{\pi} = 1/2$ into the solutions from Appendix A, and taking the limit as $\eta(\underline{\theta}) \to 0^+$, we obtain the following solutions:

$$\frac{\alpha + b_1}{\alpha} = \frac{1}{2}\frac{\rho}{\rho - \underline{\theta} + \underline{\pi}\kappa}, \qquad x_1(\bar{\theta}, \bar{m}, \emptyset) = \alpha\rho\frac{1}{2}\frac{\bar{\theta} - \underline{\theta}}{\rho - \underline{\theta} + \underline{\pi}\kappa}, \qquad x_1(\underline{\theta}, \underline{m}, \underline{\sigma}) = \alpha\rho$$

$$x_1(\underline{\theta}, \underline{m}, \emptyset) = \alpha\rho\frac{-(\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa) + \sqrt{(\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa)^2 + \frac{1}{2}(\bar{\theta} - \underline{\theta})(\rho - \underline{\theta} + \underline{\pi}\kappa)}}{\rho - \underline{\theta} + \underline{\pi}\kappa}$$

$$x_1(\underline{\theta}, \underline{m}, \bar{\sigma}) = 0, \qquad \lambda_1 = \frac{1}{\alpha\rho}, \qquad \mu_1 = \frac{1}{2}\frac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\rho - \underline{\theta} + \underline{\pi}\kappa}. \tag{27}$$

$$\frac{\alpha + b_0}{\alpha} = \frac{\rho(\mathbb{E}(\theta) - \rho)}{(\bar{\theta} - \rho)(\rho - \underline{\theta})}, \qquad x_0(\bar{m}, \emptyset, \bar{\theta}) = \alpha\rho\frac{1}{2}\frac{\bar{\theta} - \underline{\theta}}{\rho - \underline{\theta}}, \qquad x_0(\underline{\theta}, \underline{m}, \underline{\sigma}) = \alpha\rho$$

$$x_0(\underline{\theta}, \underline{m}, \emptyset) = \alpha\rho\,\frac{1}{2}\frac{\bar{\theta} - \underline{\theta}}{\bar{\theta} - \rho}, \qquad x_0(\underline{\theta}, \underline{m}, \bar{\sigma}) = 0, \qquad \lambda_0 = \frac{1}{\alpha\rho}, \qquad \mu_0 = \frac{1}{2}\frac{\mathbb{E}(\theta) - \rho}{\rho - \underline{\theta}}. \tag{28}$$

The bang-bang result occurs when efficient contracts 'jump' between private information and standard debt contracts, where both are local maxima. To focus on these bang-bang results, we first set κ such that the always audit and private information contracts provide equal expected utility ($\mathbb{E}\mathcal{U}(x_1) = \mathbb{E}\mathcal{U}(x_0)$). Solving for κ yields

$$\underline{\pi}\kappa = 2\,\frac{(\rho - \underline{\theta})(\mathbb{E}(\theta) - \rho)}{\bar{\theta} - \underline{\theta}} \tag{29}$$

It is useful to define two new parameters, one representing the excess return in good states and the other the shortfall in bad states. Thus, let $\bar{\phi}, \underline{\phi} \in \mathbb{R}^+$, where $\bar{\phi} = \bar{\theta} - \rho$, and $\underline{\phi} = \rho - \underline{\theta}$. All else equal, the entrepreneur would prefer a project with large $\bar{\phi}$, and small $\underline{\phi}$. Note that assumptions 1 and 2 require that $\underline{\phi} \in (0, \bar{\phi})$. Substituting equation 29 into the solutions for the always standard debt contract (27), we can re-write allocations as follows:

$$b_1 = \frac{\alpha\rho}{4} : \frac{\bar{\phi} + \underline{\phi}}{\bar{\phi}\underline{\phi}} - \alpha, \qquad x_1(\bar{\theta}, \bar{m}, \emptyset) = \frac{\alpha\rho}{4}\frac{(\bar{\phi} + \underline{\phi})^2}{\bar{\phi}\underline{\phi}},$$

$$x_1(\underline{\theta}, \underline{m}, \emptyset) = \frac{\alpha\rho}{4}\left[\frac{-(\bar{\phi} - \underline{\phi})^2 + \sqrt{(\bar{\phi} - \underline{\phi})^4 + 4\bar{\phi}\underline{\phi}(\bar{\phi} + \underline{\phi})^2}}{\bar{\phi}\underline{\phi}}\right], \qquad \mu_1 = \frac{1}{8}\frac{(\bar{\phi} - \underline{\phi})^2}{\bar{\phi}\underline{\phi}} \tag{30}$$

Now, consider the trade-off characterised by the first order condition for auditing q at the always audit contract. After substituting equations 29 and 30 into the first order condition for q (equation **??**), we obtain

$$\begin{aligned}
\mathcal{L}_q = {} & \frac{1}{2}\log\left[\frac{4\bar{\phi}\underline{\phi}}{-(\bar{\phi} - \underline{\phi})^2 + \sqrt{(\bar{\phi} - \underline{\phi})^4 + 4\bar{\phi}\underline{\phi}(\bar{\phi} + \underline{\phi})^2}}\right] \\
& - \frac{1}{2}\left[1 - \frac{-(\bar{\phi} - \underline{\phi})^2 + \sqrt{(\bar{\phi} - \underline{\phi})^4 + 4\bar{\phi}\underline{\phi}(\bar{\phi} + \underline{\phi})^2}}{4\bar{\phi}\underline{\phi}}\right] - \frac{1}{4}\frac{(\bar{\phi} - \underline{\phi})}{\bar{\phi}} \\
& + \frac{1}{8}\frac{(\bar{\phi} - \underline{\phi})^2}{\bar{\phi}\underline{\phi}}\log\left(1 + \frac{-(\bar{\phi} - \underline{\phi})^2 + \sqrt{(\bar{\phi} - \underline{\phi})^4 + 4\bar{\phi}\underline{\phi}(\bar{\phi} + \underline{\phi})^2}}{(\bar{\phi} + \underline{\phi})^2}\right) - \nu_1.
\end{aligned} \tag{31}$$

The first term on the right hand side of equation 31 is the welfare gain attained through auditing by verifying low type agents' reports. In this example, agents clearly prefer to be audited, $\mathcal{X}(\underline{m}, \underline{\sigma}, \underline{\theta}) > \mathcal{X}(\underline{m}, \emptyset, \underline{\theta})$ for all values of $\bar{\phi}, \underline{\phi}$. The second term captures the resource cost of this increase in consumption for low type agents whose reports are verified. The third term represents the extra resource costs expended by the intermediary in conducting more audits. The fourth term captures welfare gains attained by relaxing the incentive compatibility constraint: auditing with a higher probability directly increases the likelihood that misreporting high type entrepreneurs will be punished. The final term on the right hand side is the Lagrange multiplier capturing the shadow cost of the natural upper bound of one attached to the audit probability.

Let $\underline{\phi} \to 0^+$. By l'Hôpital's rule,

$$\lim_{\underline{\phi}\to 0^+} \frac{-(\bar{\phi}-\underline{\phi})^2 + \sqrt{(\bar{\phi}-\underline{\phi})^4 + 4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}{4\bar{\phi}\underline{\phi}} = \frac{1}{2}, \qquad \text{and}$$

$$\lim_{\underline{\phi}\to 0^+} \frac{1}{\underline{\phi}} \log\left(1 + \frac{-(\bar{\phi}-\underline{\phi})^2 + \sqrt{(\bar{\phi}-\underline{\phi})^4 + 4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}{(\bar{\phi}+\underline{\phi})^2}\right) = \frac{2}{\bar{\phi}}.$$

Substituting these results into 31 while retaining the same ordering of terms yields

$$\lim_{\underline{\phi}\to 0^+} \mathcal{L}_q(q=1) = \frac{1}{2}\log 2 - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \nu_1. \tag{32}$$

The Kuhn-Tucker multiplier ν_1 is positive. At the margin, the benefits of additional audits outweigh the costs. The first two terms show that the utility benefits accruing to low type entrepreneurs from verification of their reports exceeds the resource cost associated with awarding more low type entrepreneurs with the post-verification consumption bundle. The resource cost of audits is the product of the Lagrange multiplier on the resource constraint, total assets devoted to the project and audit costs. Here, as the downside shortfall $\underline{\phi}$ approaches zero, borrowing and assets devoted to the project are unbounded above. When downside risk is low, the benefits of auditing are small. The audit cost that equates the expected welfare of always audit and private information contracts is vanishing $\underline{\pi}\kappa \to 0^+$. The resource cost of the marginal audit is 1/4, which in this case is equal and opposite to the benefit attained from the marginal audit by relaxing the incentive compatibility constraint.

For the same case, letting $\underline{\phi} \to 0^+$, now consider the corresponding private information contract. The first order condition for q can be written as follows:

$$\mathcal{L}_q = \frac{1}{2}\log\frac{2\bar{\phi}}{\bar{\phi}+\underline{\phi}} - \frac{1}{2}\left[1 - \frac{1}{2}\frac{\bar{\phi}+\underline{\phi}}{\bar{\phi}}\right] - \frac{1}{2}\frac{(\bar{\phi}-\underline{\phi})^2}{\bar{\phi}(\bar{\phi}+\underline{\phi})} + \frac{1}{4}\frac{\bar{\phi}-\underline{\phi}}{\underline{\phi}}\log\left(\frac{\underline{\phi}}{\bar{\phi}-\underline{\phi}}+1\right) + \nu_0.$$

Taking the limit as $\underline{\phi} \to 0^+$ yields

$$\lim_{\underline{\phi}\to 0^+} \mathcal{L}_q(q=0) = \frac{1}{2}\log 2 - \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + \nu_0, \tag{33}$$

The first term captures the direct welfare benefit from verifying entrepreneur reports and providing entrepreneurs with the consumption bundle $x_0(\underline{\theta}, \underline{m}, \underline{\sigma}) > x_0(\underline{\theta}, \underline{m}, \emptyset)$. This benefit, and the resource cost associated with it and captured in the second term, are identical to those in the always audit contract (32). This is due to the fact that these consumption bundles are identical in both contracts for this limiting case: low type agents whose reports are verified $(\mathcal{X}(\underline{\theta}, \underline{m}, \underline{\sigma}))$ consume $\alpha\rho$ and low type agents whose reports are unverified $(\mathcal{X}(\underline{\theta}, \underline{m}, \emptyset))$ consume $\alpha\rho/2$ in both contracts. As in the always audit contract (32), the fourth term capturing the relaxation in the incentive compatibility constraint is equal to $1/4$.

The third term, capturing the cost of the marginal audit, is greater in magnitude than under the standard debt contract. Audit costs as a fraction of assets employed in the project are constant across contracts by assumption, yet leverage in the non-contingent debt contract is greater than in the standard debt contract. We can see this by taking the limit of the ratio of assets devoted to the project in the two contracts: $\lim_{\underline{\phi}\to 0} \dfrac{\alpha + b_0}{\alpha + b_1} = 2$. ■

B STANDARD DEBT CONTRACTS WITH TYPE-II ERRORS

Assume that the probability of Type-II errors $\eta(\bar{\theta})$ following the audit of a high type entrepreneur is low, and that the optimal contract is standard debt $(q^* = 1)$. We can find approximate closed form solutions to optimal contracts using a first order Taylor expansion of the Incentive Compatibility Constraint around $\eta(\bar{\theta}) = 0$. Denote the efficient contract at $\eta(\bar{\theta}) = 0$ with all other parameters constant by Γ_0, with associated public actions and allocations labelled $b_0, q_0 = 1, \mathcal{X}_0, z_0$.

$$
\begin{aligned}
U(\mathcal{X}_0(\bar{\theta}, \bar{m}, \emptyset)) \geq U(\mathcal{X}_0(\bar{\theta}, \underline{m}, \bar{\sigma})) &+ U'(\mathcal{X}_0(\bar{\theta}, \underline{m}, \bar{\sigma}))[\mathcal{X}(\bar{\theta}, \underline{m}, \bar{\sigma}) - \mathcal{X}_0(\bar{\theta}, \underline{m}, \bar{\sigma})] \\
&- U'(\mathcal{X}_0(\bar{\theta}, \bar{m}, \emptyset))[\mathcal{X}(\bar{\theta}, \bar{m}, \emptyset) - \mathcal{X}_0(\bar{\theta}, \bar{m}, \emptyset)] \\
&+ \eta(\bar{\theta})[U(\mathcal{X}_0(\bar{\theta}, \underline{m}, \underline{\sigma})) - U(\mathcal{X}_0(\bar{\theta}, \underline{m}, \bar{\sigma}))]
\end{aligned}
$$

Note that $\mathcal{X}_0(\bar{\theta}, \bar{m}, \emptyset) = \mathcal{X}_0(\bar{\theta}, \underline{m}, \bar{\sigma})$, which combined with the budget constraints allows us to simplify the above expression as follows:

$$
\mathcal{Z}(\underline{m}, \bar{\sigma}) - \mathcal{Z}(\bar{m}, \emptyset) \geq \frac{\eta(\bar{\theta})}{U'(\mathcal{X}_0(\bar{\theta}, \underline{m}, \bar{\sigma}))}[U(\mathcal{X}_0(\bar{\theta}, \underline{m}, \underline{\sigma})) - U(\mathcal{X}_0(\bar{\theta}, \underline{m}, \bar{\sigma}))]. \tag{34}
$$

When utility is log, we can solve directly using Proposition 6. After rearranging and simplifying, our first order approximation of the ICC can be written as a linear expression in terms of contracted repayments:

$$
\mathcal{Z}(\underline{m}, \bar{\sigma}) - \mathcal{Z}(\bar{m}, \emptyset) \geq \eta(\bar{\theta}) \, \frac{\alpha\rho}{1-\zeta} \log\left[1 + \frac{\bar{\pi}\zeta(1-\bar{\pi}\zeta)}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right] \tag{35}
$$

This linear approximation to the ICC permits closed form approximations to allocations and leverage for efficient contracts, which are not presented here.

One interpretation of equation 35 is that it specifies a small non-refundable fee paid by all entrepreneurs who declare a low type return. Entrepreneurs whose reports are overturned by the audit signal would be required to repay the full contracted repayment $\mathcal{Z}(\bar{m}, \emptyset)$ in addition to the small fee. When $\eta(\bar{\theta})$ is small, this fee is negligible.

C FIGURES

Figure 1: Efficient contracts, project risk and audit costs. $(U(x) = \sqrt{x}, \rho = 1, \mathbb{E}(\theta) = 1.2, \bar{\pi} = 0.9, \eta(\underline{\theta}) = 0.01)$.

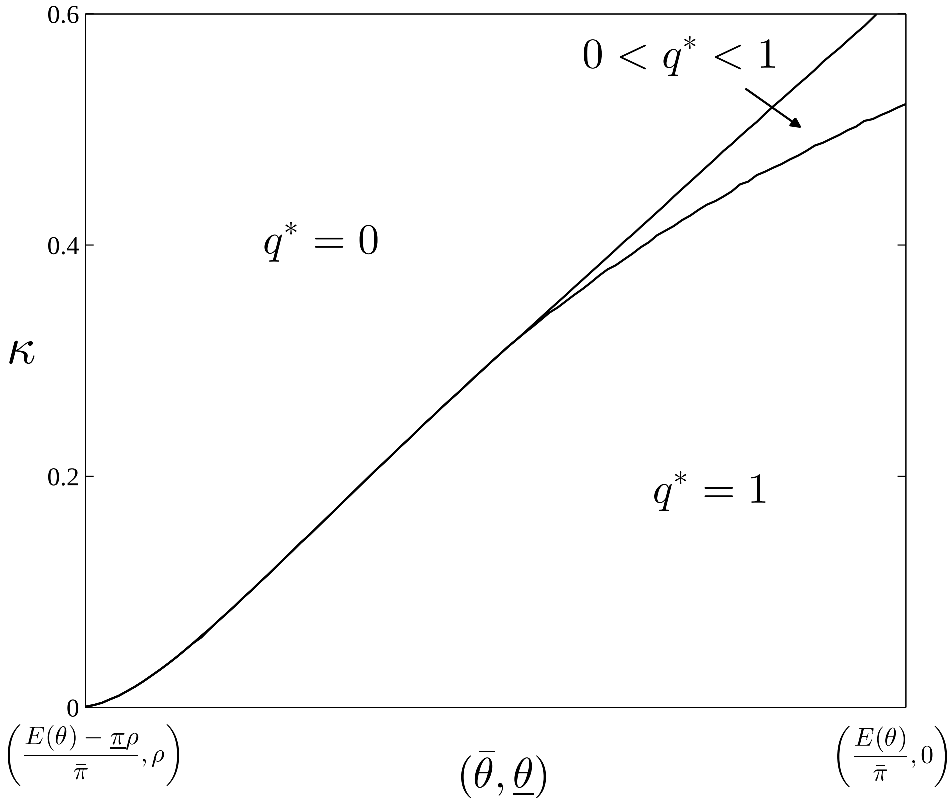


Figure 2: The determination of optimal contracts when there are multiple local maxima. $(U(x) = \log x, \rho = 1, \mathbb{E}(\theta) = 1.2, \bar{\pi} = 0.9, (\bar{\theta} - \underline{\theta}) = 0.3, \eta(\underline{\theta}) = 10^{-4}, \alpha = 1, \kappa \approx 0.18)$

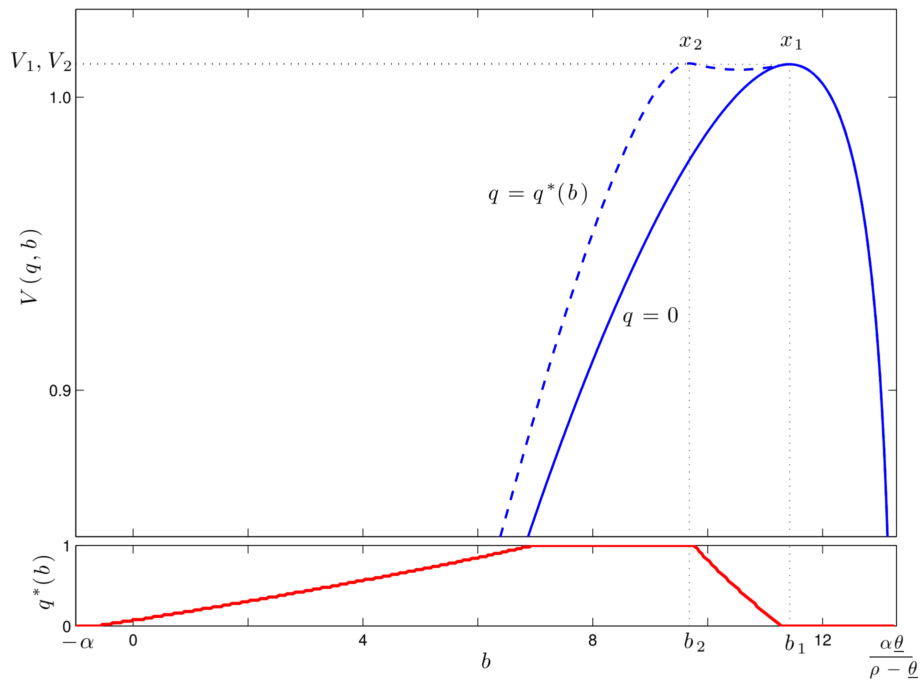
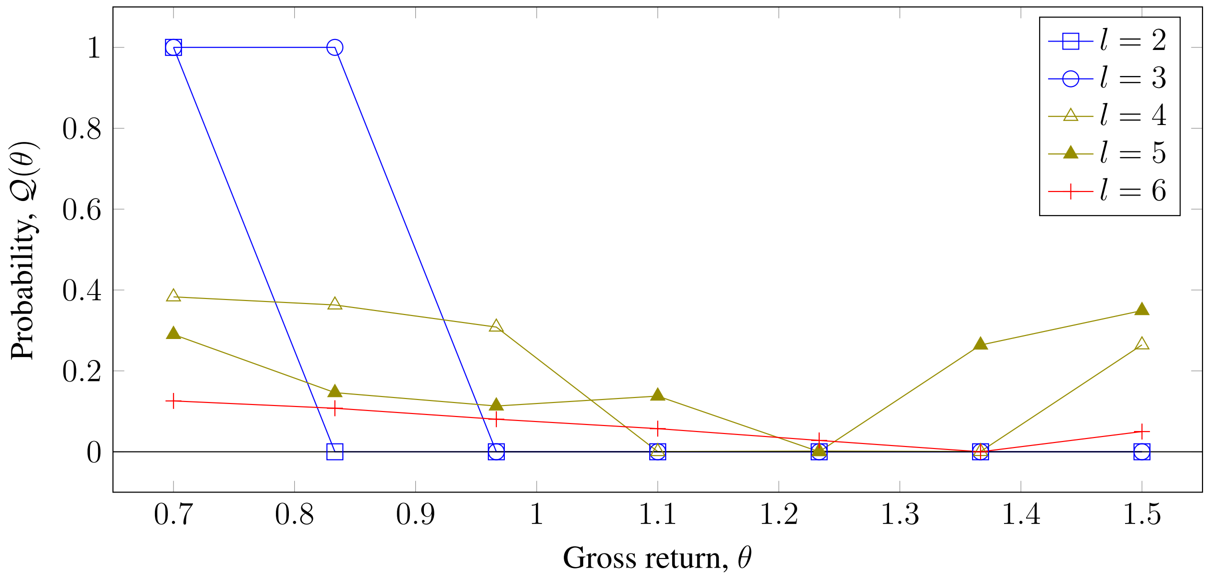
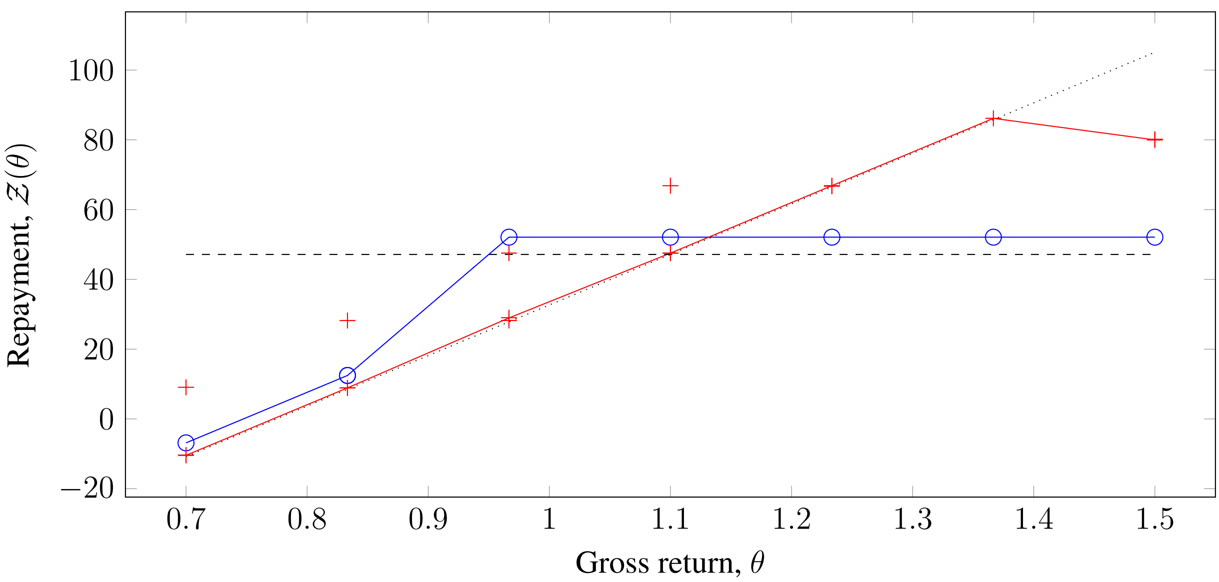


Figure 3: Numerical example

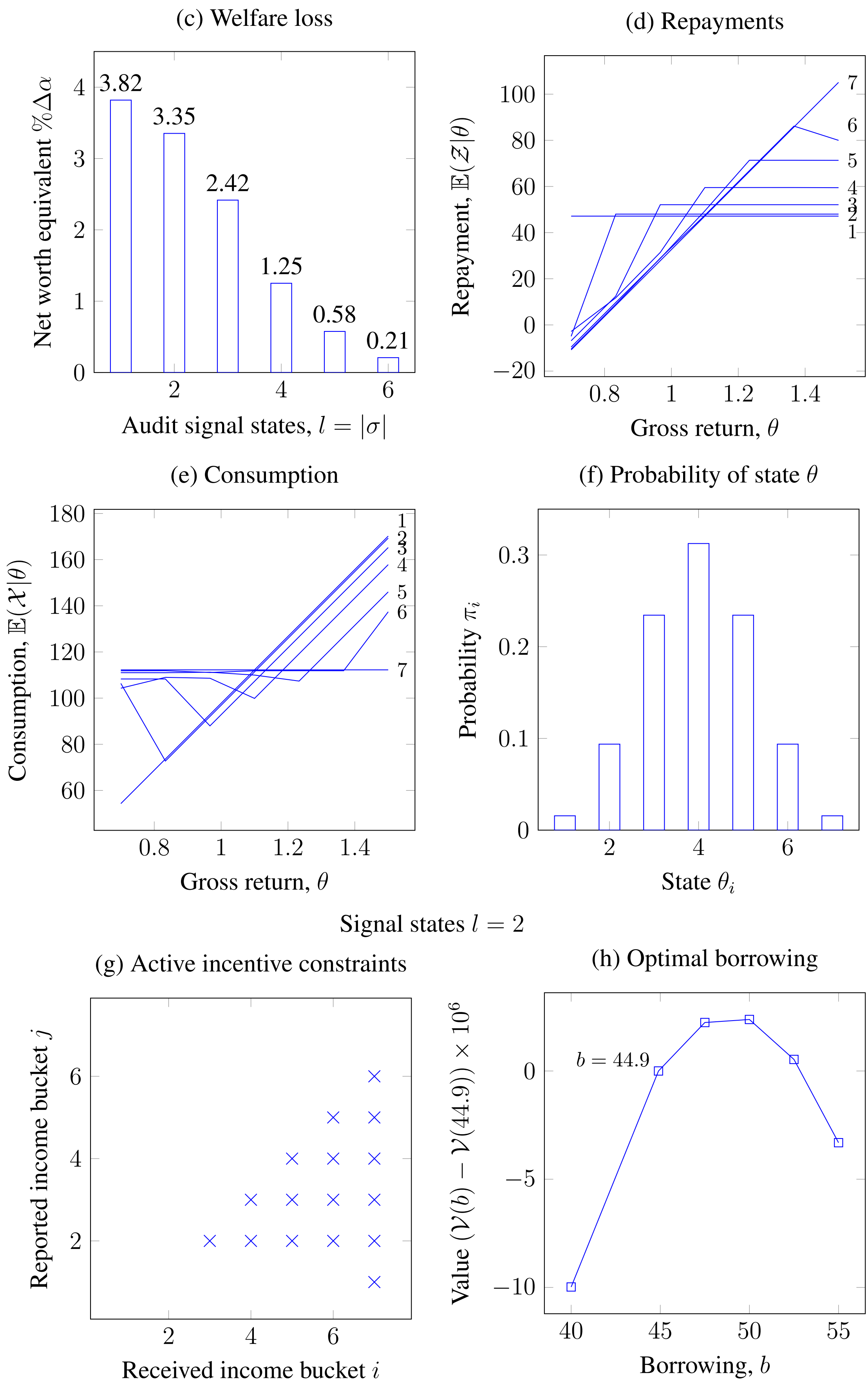
(a) Audit strategy, $\mathcal{Q}$



(b) Repayments, $\mathcal{Z}(\theta)$



Key: Positive probability repayments are plotted as marks (o for $l = 3$; + for $l = 7$). Conditional expectations of repayments $\mathbb{E}(\mathcal{Z}|\theta)$ are plotted as solid lines. Dashed lines represent repayments for the non-contingent and first best contracts.


(c) Welfare loss
Net worth equivalent %$\Delta\alpha$
3.82
3.35
2.42
1.25
0.58
0.21
Audit signal states, $l = |\sigma|$
(d) Repayments
Repayment, $\mathbb{E}(\mathcal{Z}|\theta)$
Gross return, θ
(e) Consumption
Consumption, $\mathbb{E}(\mathcal{X}|\theta)$
Gross return, θ
(f) Probability of state θ
Probability π_i
State θ_i
Signal states $l = 2$
(g) Active incentive constraints
Reported income bucket j
Received income bucket i
(h) Optimal borrowing
Value $(\mathcal{V}(b) - \mathcal{V}(44.9)) \times 10^6$
$b = 44.9$
Borrowing, b