Competitive Fair Division under linear preferences

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1 Introduction

The fair division of a bundle of goods (*manna*) among agents with heterogenous preferences is an important challenge for normative economic analysis. The difficulty is to combine efficiency (Pareto optimality) with some convincing notion of *fairness.* When agents are responsible for their ordinal preferences, and cardinal measures of utility are not relevant, the division rule favored by economist is the *Competitive equilibrium with equal incomes*, thereafter CEEI, invented almost 50 years ago (Varian, Kolm).

Here we discuss this rule in the important special case where individual preferences are *linear.* This means that the goods are perfect substitutes and each participant in the division needs only report $p-1$ rates of substitution if there are p goods to share. This assumption is less drastic than may appear.

On the one hand it is the only practical approach in several free web sites (Adjusted Winner, Spliddit) computing fair solutions for concrete division problems: siblings are sharing family heirlooms, partners divide the common assets upon dissolving their partnership, and so on. They are asked to distribute 100 points over the (divisible) goods, and the resulting distribution is interpreted as an additive utility function representing the underlying linear preferences. Anything more sophisticated involving comparisons of subsets of goods is impractical when we have more than a couple of goods. On the other hand perfect substitutability is realistic when we divide inputs into a production process such as land, machines with same function but different specifications, hours of work with different skills, computing resources, etc..

One critical property of the CEEI rule when preferences are linear (or more generally homothetic) is that it maximizes the product of the canonical linear (homothetic) utility functions over all feasible allocations (Gale). Therefore the

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corresponding profile of utilities is unique, and depends continuously upon the parameters of the problem (MRSs and endowment of the goods).[1]

Another fair division method is also popular, in fact more so on the free websites just mentioned: the *Relative Egalitarian* solution (aka Egalitarian Equivalent, Adjusted Winner), thereafter REG, simply equalizes the relative gains of the participants, i. e., the number of points on each individual share.[2] By contrast the CEEI solution offers equal opportunity of choice, because everyone gets his best allocation in the common budget set: in particular no one *envies* the share of anyone else.

We show here that the CEEI division rule meets two additional properties that the REG rule fails: one is the familiar *Resource Monotonicity* property (RM), the other is a new incentive compatibility property that we call *Verification-proofness* (VP). In fact (a group version of) the VP property, together with standard properties of efficiency and symmetry, delivers a compact *characterization* of the CEEI solution.

RM says that as we add new goods to the pot (or increase the quantity of some goods), every agent weakly benefit from the change. This appealing normative property has an incentive interpretation as well: if RM fails an agent will sometime prefer to waste some goods that adding them to the common pot. The CEEI rule fails RM on the general Arrow Debreu domain[3]. The REG rule fails it even on the linear domain.

Recall the*Strategyproofness* requirement that no agent should benefit by reporting anything but her true preferences to the division manager. Even on the linear domain, no efficient division rule can be both strategyproof and minimally fair (e.g., treating participants symmetrically: Zhou, Schummer). The VP property weakens *Strategyproofness* by distinguishing the goods that our agent actually consumes upon implementing the rule, from those she doesn't consume. We assume that preferences over individual allocations are *ex post verifiable*: because you consume positive amounts of goods a and b, I can observe your marginal rate of substitution between a and b. This assumption is not convincing in the case of family heirlooms, much more so when we share inputs such as land or machines of which the productivity is easily inspected.

Ex post verifiability allows the manager to punish misreports over goods actually consumed, but an agent can still lie about the goods he will not consume at all. Verification-proofness makes such misreport unattractive to each participant.

It is easy to manipulate the REG division rule by exagerating the worth of those goods we do not get (as long as we still don't get them after the misreport). Not so for the CEEI rule, that even meets a *group version* of VP: no joint misreport where everyone is truthful about goods he actually consumes, can benefit the coalition members, all weakly and at least one strictly.

[1] In the general Arrow Debreu domain of preferences, multiple competitive solutions can happen, and the selction problem has no easy normative answer.

[2] The REG solution is an instance of the familiar Kalai-Smorodinski bargaining solution.

[3] And so does any efficient rule guaranteeing a fair share of the pot to every participant: Moulin and Thomson.

Out main result (Theorem 1) is that the CEEI rule is the only efficient and symmetric division rule that is Group-VP and treats two goods on which all individual MRSs agree as a single object.

2 The model

The finite set of agents is N with generic element i. We assume $|N| = n \geq 2$. The finite set of (divisible) goods is A with generic element a. The manna consists of 1 unit of each object.

Agent i's allocation (or share) is some $z_i \in [0,1]^A$; the profile $z = (z_i)_{i \in N}$ is a feasible allocation if $\sum_N z_i = e^A$, where all coordinates of e^A in $\mathbb{R}_+^A$ are 1. The set of feasible allocations is $\Phi(N, A)$.

Each agent is endowed with linear preferences over $[0,1]^A$, represented for convenience by a vector $u_i \in \mathbb{R}_+^A$ (a utility function). We keep in mind that only the *ordinal* preferences matter, i. e., for any $\lambda > 0$, u_i and λu_i carry the same information. Given a profile of shares z we write i's corresponding utility as $U_i = u_i \cdot z_i = \sum_A u_{ia} z_{ia}$.

A **division problem** is a triple (N, A, u) and the corresponding set of feasible utility profiles is $\Psi(N, A, u)$. Note that we may have *useless* goods ($u_{ia} = 0$ for all i) or *uninterested* agents ($u_{ia} = 0$ for all i). Otherwise we speak of *useful* goods and *interested* agents.

We give two equivalent definitions of a division rule, in terms of utility profiles or of feasible allocations. When we rescale each u_i as $\lambda_i u_i$ the new profile of utilities is written $\lambda * u$.

Definition 1
i) A division rule F associates to every problem (N, A, u) a utility profile $F(N, A, u) = U \in \Psi(N, A, u)$. Moreover $F(N, A, \lambda * u) = \lambda * U$ for any rescaling λ where $\lambda_i > 0$ for all i.

ii) A division rule f associates to every problem (N, A, u) a subset $f(N, A, u)$ of $\Phi(N, A)$ such that for some $U \in \mathbb{R}_+^A$:

$$f(N, A, u) = \{z \in \Phi(N, A) | (u_i \cdot z_i)_{i \in N} = U\}$$

Moreover $f(N, A, \lambda * u) = f(N, A, u)$ for any rescaling λ where $\lambda_i > 0$ for all i.

The one-to-one mapping from F to f is clear. Definition 1 makes no distinction between two allocations with identical welfare consequences.

Note that a useless good can be divided arbitrarily, and, if the rule is efficient, an uninterested agent can only consume positive amounts of useless goods.

A Lemma about efficient allocations For $z \in \Phi(N, A)$ define the bipartite $N \times A$ graph $\Gamma(z) = \{(i, a) | z_{ia} > 0\}$.

Define the set $\Psi^{eff}(N, A, u)$ of efficient utility profiles.

Lemma 1 *If $U \in \Psi^{eff}(N, A, u)$ then there is some $z \in \Phi(N, A)$ representing U such that $\Gamma(z)$ is a forest (acyclic graph).*

Proof. Assume first that all goods are useful. Pick z representing U and assume a K-cycle in $\Gamma(z)$: $z_{ka_k}, z_{ka_{k-1}} > 0$ for $k = 1, \cdots, K$, where $z_{1a_{1-1}} = z_{ia_K}$. Then $u_{ka_k}, u_{ka_{k-1}}$ are positive for all k: if $u_{ka_k} = 0$ efficiency and $u_{Na_k} > 0$ imply $z_{ka_k} = 0$.
Assume now

$$\frac{u_{1a_1}}{u_{2a_1}} \cdot \frac{u_{2a_2}}{u_{3a_2}} \cdot \cdots \cdot \frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}} \cdot \frac{u_{Ka_K}}{u_{1a_K}} > 1 \tag{1}$$

Then we can pick arbitrarily small positive numbers ε_k such that

$$\frac{u_{1a_1} \cdot \varepsilon_1}{u_{1a_K} \cdot \varepsilon_K}, \frac{u_{2a_2} \cdot \varepsilon_2}{u_{2a_1} \cdot \varepsilon_1}, \cdots, \frac{u_{Ka_K} \cdot \varepsilon_K}{u_{Ka_{K-1}} \cdot \varepsilon_{K-1}} > 1$$

and the corresponding transfer to each agent k of ε_k units of good k against ε_{k-1} units of good $k-1$ is a Pareto improvement, contradiction. Therefore (1) is impossible; the opposite strict inequality is similarly ruled out so we conclude

$$\frac{u_{1a_1}}{u_{2a_1}} \cdot \frac{u_{2a_2}}{u_{3a_2}} \cdot \cdots \cdot \frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}} \cdot \frac{u_{Ka_K}}{u_{1a_K}} = 1$$

Now if we perform a transfer as above where

$$\frac{u_{1a_1} \cdot \varepsilon_1}{u_{1a_K} \cdot \varepsilon_K} = \frac{u_{2a_2} \cdot \varepsilon_2}{u_{2a_1} \cdot \varepsilon_1} = \cdots = \frac{u_{Ka_K} \cdot \varepsilon_K}{u_{Ka_{K-1}} \cdot \varepsilon_{K-1}} = 1$$

the utility profile U is unchanged.If we choose the numbers ε_k as large as possible for feasibility, this will bring at least one entry (k, a_k) or (k, a_{k-1}) to zero, so in our new representation z' of U the graph $\Gamma(z')$ has fewer edges. We can clearly repeat this operation until we eliminate all cycles of $\Gamma(z)$.

Now if some goods are useless we give them all to an arbitrary agent and the statement still holds. ■

3 The CEEI rule

The set $f^{CEEI}(N, A, u)$ contains all competitive allocations when agents are endowed with equal incomes (or equal shares of all goods): the allocation z is a *Competitive Equilibrium with Equal Incomes* (CEEI) if there is a price $p \in \mathbb{R}^A_+$ such that $\sum_A p_a = n$ and

$$z^i \in \arg\max_{y^i \in \mathbb{R}^A_+} \{u^i \cdot y^i | p \cdot y^i \leq 1\} \text{ for all } i$$

Theorem *(Gale 1960): the CEEI rule f^{CEEI} selects precisely all the feasible allocations in the economy (N, A, u) maximizing the product of utilities:*

$$f^{CEEI}(N, A, u) = \arg\max_{\Phi(N,A)} \Pi_N u^i \cdot z^i$$

This is a bona fide division rule in the sense of Definition 1: the corresponding utility profile $(u^i \cdot z^i)_{i \in N} = F^{CEEI}(N, A, u)$ is unique.
The main consequence of this remarkable result is that at an MP allocation, the participants have *equal opportunities* in the sense of the familiar property

No Envy: $u^i \cdot z^i \geq u^i \cdot z^j$ for all i, j

the KKT characterization of F^{CEEI} The CEEI rule maximizes a quasi-concave function over a convex compact set, hence it is characterized by a system of first order conditions often called the *KKT conditions*. They play a big role throughout the paper.

We assume $u_{Na} > 0$ for all a and $u_{iA} > 0$ for all i. Then $U = F^{CEEI}(N, A, u)$ has $U_i > 0$ for all i.

Lemma 2 *Fix a problem (N, A, u), a profile $U \in \Psi(N, A, u)$, and let A^* be the set of useful goods. Then $U = F^{CEEI}(N, A, u)$ if and only if U is achieved by an allocation $z \in \Phi(N, A^*)$ such that $\Gamma(z)$ is a forest where a node i is isolated iff if i is uninterested (then $z_i = 0$), and among interested agents we have*

$$\frac{u_{ia}}{U_i} \geq \frac{u_{ja}}{U_j} \text{for all } (i, a) \in \Gamma(z) \text{ and all } j \in N \tag{2}$$

(in particular $U_i > 0$)

Proof: It is clearly enough to prove the result when all goods are useful and all agents interested.

only if. Take z representing U s. t. $\Gamma(z)$ is a forest (Lemma 1). Then if $z_{ia} > 0$ we can transfer some small amount of a to any agent j, and the inequality above makes sure this does not increases the Nash product.

if. We check that the system $\{\frac{u_{ia}}{U_i} \geq \frac{u_{ja}}{U_j} \text{for all } (i, a) \in \Gamma(z) \text{ and all } j\}$ is precisely the KKT one. The Lagrangien is

$$\mathcal{L}(z, \delta, \theta) = \sum_N \ln(u_i \cdot z_i) - \sum_A \delta_a(z_{Na} - 1) + \sum_{i,a} \theta^+_{ia} z_{ia} + \theta^-_{ia}(1 - z_{ia})$$

where $\theta \geq 0$ and the sign of each δ_a is arbitrary. The conditions $\frac{\partial \mathcal{L}}{\partial z}(z, \delta, \theta) = 0$ amount to

$$\frac{u_{ia}}{U_i} - \delta_a + \theta^+_{ia} - \theta^-_{ia} = 0 \text{ for all } i, a \tag{3}$$

If $z_{ia} = 1$ then $z_{ja} = 0$ for all $j \neq i$, and system (3) gives $\frac{u_{ia}}{U_i} \geq \delta_a \geq \frac{u_{ja}}{U_j}$. If $0 < z_{ia} < 1$ then $\frac{u_{ia}}{U_i} = \delta_a$, and for another agent j we have $z_{ja} < 1$ hence $\frac{u_{ja}}{U_j} \leq \delta_a$.■

4 Resource Monotonicity

The following property has played a major role in the modern fair division literature (see surveys by Thomson):

- **Resource Monotonicity**: $A \subset B \Longrightarrow U^i(N, A, u_{[A]}) \leq U^i(N, B, u_{[B]})$

This simply says that more goods to divide cannot be bad news to anyone, so that no one has an incentive to sabotage the discovery of additional "manna". However recall that in the general Arrow-Debreu preference domain, no efficient division rule can be resource monotonic and meet at the same time the oldest test of the cake division literature: every participant can claim his fair share of the resources and be gone (Moulin and Thomson 1988).

- **Fair Share Guarantee:** $U_i = u_i \cdot z_i \geq u_i \cdot (\frac{1}{|N|}e^A)$

In the linear domain however, this incompatibility disappears:

Lemma 3 *The division rule F^{CEEI} is resource monotonic.*

Proof We first generalize the definition of F^{CEEI}, f^{CEEI} to problems where the endowment ω_a of each good is arbitrary, and we check that the KKT conditions capturing the optimal allocations $f^{CEEI}(N, A, \omega, u)$ are unchanged. Then we fix N, A, u, ω, ω' such that $\omega \leq \omega'$. For $\lambda \in [0,1]$ we write $\omega^\lambda = (1-\lambda)\omega + \lambda\omega'$, and for every forest Γ in $N \times A$ we define

$$\mathcal{B}(\Gamma) = \{\lambda \in [0,1] | \exists z \in f^{CEEI}(N, A, \omega^\lambda, u) : \Gamma(z) = \Gamma\}$$

Note that $\mathcal{B}(\Gamma)$ can be empty or a singleton, but if it is not, then it is an interval. To see this take $z \in f^{CEEI}(\omega^\lambda), z' \in f^{CEEI}(\omega^{\lambda'})$ such that $\Gamma(z) = \Gamma(z')$. For any $\omega'' = (1-\mu)\omega^\lambda + \mu\omega^{\lambda'}$ the allocation $z'' = (1-\mu)z + \mu z'$ is feasible, $z^{''} \in \Phi(N, A, \omega'')$, the forest $\Gamma(z'')$ is unchanged, and the KKT system (2), which holds at z and z', also holds at z''. Thus $z'' \in f^{CEEI}(\omega'')$ and the claim is proven. In fact $\mathcal{B}(\Gamma)$ is a closed interval because the mapping $\omega \rightarrow U(\omega)$ is continuous (an easy consequence of Berge Theorem).

Next we claim that inside an interval $\mathcal{B}(\Gamma)$ the rule F^{CEEI} is resource monotonic. The forest Γ is a union of subtrees. If a subtree contains a single agent i, she eats (in full) the same subset of goods for any λ in $\mathcal{B}(\Gamma)$, hence her utility increases weakly in λ. If a subtree of Γ connects the subset S of agents, then system (2) fixes the direction of the utilities $(U_i)_{i \in S}$, because if i, j both consume good a we have $\frac{u_{ia}}{U_i} = \frac{u_{ja}}{U_j}$ and such equalities connect all agents in S. As λ increases in $\mathcal{B}(\Gamma)$ this implies that these U_i are co-monotonic, but as the agents in S eat together the same subset of goods, efficiency implies that the U_i increase weakly.

Finally Lemma 1 implies that the closed intervals $\mathcal{B}(\Gamma)$ cover $[0,1]$, so $[0,1]$ is a finite union of closed intervals with non empty interior, and the desired conclusion $U(\omega) \leq U(\omega')$ follows. ■

To see the force of Lemma 3, we note that the very popular *Relative Egalitarian (*REG) rule is not resource monotonic on the linear domain. This rule equalizes the normalized utilities as much as permitted by efficiency.

If $u_{ia} > 0$ for all $i \in N, a \in A$, the rule picks the utility profile $U \in \Psi^{eff}(N, A, u)$ defined by[4][5]

$$\frac{U_i}{u^i \cdot e^A} = \frac{U_j}{u^j \cdot e^A} \text{ for all } i, j$$

[4] If some u_{ia} are zero this equality may be incompatible with efficiency and we need to use a lexicographic refinment.

[5] The rule F^{REG} meets the Fair Share Guaranteed property but may generate envy.

5 Verification-Proofness

We use the notation $[z_i]$ for the support of $z_i \in [0,1]^A$, i. e., the set of goods a such that $z_{ia} > 0$. In the following definitions, we fix N, A and omit them in the argument of division rules.

Given two (individual) utilities $u_i, v_i \in \mathbb{R}_+^A$, and a subset $Y \subseteq A$, we say that v_i is truthful to u_i on Y if u_{ia} and v_{ia} are both zero or both non zero for all $a \in Y$, and $\frac{u_{ia}}{u_{ib}} = \frac{v_{ia}}{v_{ib}}$ for all $a, b \in Y$.[6]

Definition 2 *The division rule f is Verification-Proof (VP) if for all $i \in N, u \in \mathbb{R}_+^{A \times N}$ and $v_i \in \mathbb{R}_+^A$*

$$\{v^i \text{ is truthful to } u^i \text{ on } [z_i], \text{ for some } z \in f(v_i, u_{-i})\} \Longrightarrow F_i(u) \geq u_i \cdot z_i$$

We need the group version of VP in our characterization result. Fix $S \subseteq N$, and $u, v \in \mathbb{R}_+^{A \times N}$ such that $u_j = v_j$ for $j \notin S$. Given the rule f, we say that v is a *non verifiable manipulation by S at u* if

$$\{v_i \text{ is truthful to } u_i \text{ on } [z_i] \text{ for all } i \in S, \text{ for some } z \in f(v)\} \tag{4}$$

and

$$u_i \cdot z_i \geq F_i(u) \text{ for all } i \in S, \text{ with at least one strict inequality} \tag{5}$$

Definition 3 *The division rule f is GroupVerification-Proof (GVP) if no coalition ever has a non verifiable manipulation.*

Lemma 4 *The CEEI rule is GroupVerification-Proof.*

We prove in fact that CEEI meets a stronger property. Given $u, v \in \mathbb{R}_+^{A \times N}$ and $y \in F(u), z \in F(v)$

$$\{v_i \text{ is truthful to } u_i \text{ on } [z_i] \text{ for all } i\} \Longrightarrow \{u_i \cdot y_i \geq u_i \cdot z_i \text{ for all } i\} \tag{6}$$

Proof: Under the above premises we assume that the set N^* of agents such that $u_i \cdot y_i < u_i \cdot z_i$ is non empty, and derive a contradiction.

Fix $i \in N^*$ and consider an agent j, if any, such that $[z_i] \cap [y_j] \neq \varnothing$. We claim that $j \in N^*$. Pick $a \in [z_i] \cap [y_j]$, i.e., $(i, a) \in \Gamma(z)$ and $(j, a) \in \Gamma(y)$. Apply (2) first at u then at v

$$\frac{u_{ja}}{u_j \cdot y_j} \geq \frac{u_{ia}}{u_i \cdot y_i} \text{ and } \frac{v_{ia}}{v_i \cdot z_i} \geq \frac{v_{ja}}{v_j \cdot z_j}$$

By the truthfullness assumption $\frac{v_{ia}}{v_i \cdot z_i} = \frac{u_{ia}}{u_i \cdot z_i}$ and $\frac{v_{ja}}{v_j \cdot z_j} = \frac{u_{ja}}{u_j \cdot z_j}$, thus the above inequalities imply

$$\frac{u_i \cdot y_i}{u_j \cdot y_j} \geq \frac{u_{ia}}{u_{ja}} \text{ and } \frac{u_{ia}}{u_{ja}} \geq \frac{u_i \cdot z_i}{u_j \cdot z_j}$$

(note that $u_{ja} > 0$ by efficiency and $y_{ja} > 0$). Combining this with $u_i \cdot y_i < u_i \cdot z_i$ gives $u_j \cdot y_j < u_j \cdot z_j$ as claimed.

[6] With the convention that the equality holds if both denominators are null.

The claim implies that $\cup_{i \in N^*}[z_i]$ and $\cup_{j \in N \setminus N^*}[y_j]$ are disjoint. But the complement of $\cup_{j \in N \setminus N^*}[y_j]$ in A contains all the goods eaten exclusively by agents in N^* at y, while $\cup_{i \in N^*}[z_i]$ contains all the goods that are at least partially eaten by agents in N^* at z: therefore our assumption that everyone in N^* strictly prefers z to y at u contradicts the efficiency of y.■

6 Characterization of F^{CEEI}

We use four axioms in addition to GVP. The first two are standard

- **Efficiency** *(EFF): $F(N, A, u) \in \Psi^{eff}(N, A, u)$ for all (N, A, u)*
- **Symmetry** *(SYM)**:** (the label of agents and goods does not matter) F is invariant with respect to permutations of N, and of A*

The next two axioms come from the fair division context, and they are very mild.

We say that problem (N, A, u) is *partitioned* in two subproblems (N^k, A^k, u^k), $k = 1, 2$, if N^1, N^2 partition N, A^1, A^2 partition A, and agents in N^k do not care for objects in A^l, for $\{k, l\} = \{1, 2\}$.

- **Partition** *(PAR): the rule solves each subproblem of a partitioned problem separately*

(i. e., $F(N, A, u)$ is the concatenation of $F(N^k, A^k, u^k)$ for $k = 1, 2$)

We say that two goods a, b are *equivalent* in problem (N, A, u) if

$$u_{ia} \cdot u_{jb} = u_{ib} \cdot u_{ja} \text{ for all } i, j \in N$$

(So a useless good ($u_{ia} = 0$ for all i) is equivalent to every other good). To merge goods a and b means to replace the problem (N, A, u) by (N, A^*, u^*) where a, b become a single good a^* with utilities $u^*_{ia^*} = u_{ia} + u_{ib}$ for all i, while utilities for other goods are unchanged.

- **Equivalent Goods** *(EG)**:** if we merge two equivalent goods a, b then $F(N, A^*, u^*) = F(N, A, u)$*

Note that a useless good can be merged with any other good, without altering the utility profile: this is a way of saying that useless object are irrelevant.

All four axioms are met by many welfarist rules, such as the Relative Egalitarian one.[7]

Theorem: *The CEEI rule is characterized by Efficiency, Symmetry, Partition, Equivalent Goods, and Group Verification-Proofness*

We stress that the only fairness axiom in the Theorem is Symmetry. No Envy is not used.

[7] Other instances include, for any $q, -\infty \leq q \leq +\infty$, the rule minimizing $sign(q) \sum_N (\frac{u_i}{u_i \cdot e^A})^q$.

Proof

The CEEI rule clearly meets the four axioms just introduced.

Conversely we fix a rule F, f meeting all axioms. The set N is fixed throughout and written $[n] = \{1, \cdots, n\}$. The set A varies and in view of the Symmetry property is always written $[m]$. Then a problem is described by a non negative $n \times m$ *utility matrix* $u = [u_{ia}]_{i \in [n], a \in [m]}$. We write $\mathcal{R}(n, m)$ the set of such matrices, and $\mathcal{A}(n, m)$ the set of *allocation matrices* $z = [z_{ia}]$ defined by: $z \in \mathcal{R}(n, m)$ and $\sum_i z_{ia} = 1$ for all a. The rule F maps $\mathcal{R}(n, m)$ into $\mathbb{R}_+^{[n]}$ and f is a correspondence into $\mathcal{A}(n, m)$. Recall from Definition 1 that F, f are *scale invariant*: multiplying a certain row of u by a (strictly) positive constant does not change the image of f, and multiplies the corresponding coordinate of $F(u)$ by the constant.

Step 1 *Fix $m, a \in [m]$ and $i \in [n]$. Consider two problems $u, u' \in \mathcal{R}(n, m)$ that only differ in that $u'_{ia} = 0 < u_{ia}$. Then*

$$\{z_{ia} = 0 \text{ for some } z \in f(u)\} \Longrightarrow f(u) = f(u')$$

This follows from GroupVP. There is some $j, j \neq i$, who likes a otherwise by efficiency i should be eating all a at u. Then efficiency at u' implies $z'_{ia} = 0$ for any $z' \in f(u')$. Pick z as in the premises above: VP (Definition 2) from the truth u to the misreport u' gives $u_i \cdot z_i \geq u_i \cdot z'_i$; and applied from u' to u it gives $u'_i \cdot z'_i \geq u'_i \cdot z_i$, so we get $u_i \cdot z_i = u_i \cdot z'_i$. Now fix another agent j and apply GroupVP from u to u', where both i and j are truthful on $[z'_i]$ and $[z'_j]$ respectively: because $u_i \cdot z_i = u_i \cdot z'_i$ we must have $u_j \cdot z_j \geq u_j \cdot z'_j$. And GroupVP from u' to u gives the opposite inequality. We see that z and z' yield the same utilities at u; as z' is arbitrary in $f(u')$ we get $f(u') = f(u)$, and the desired equality by Definition 1.

Step 2 For each $\ell \in [n]$ and $w \in \mathbb{R}_+^{[n]}$ we define the following matrix $u^{\ell,w} \in \mathcal{R}(n, n+1)$, with the help of the familiar symbol δ : $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$:

$$u_{ij}^{\ell,w} = \delta_{ij} w_i \text{ for all } i, j \leq n \tag{7}$$

$$u_{i(n+1)}^{\ell,w} = 1 \text{ if } i \leq \ell \text{ ; } u_{i(n+1)}^{\ell,w} = 0 \text{ if } i \geq \ell + 1 \tag{8}$$

and we write $\mathcal{D}(n, n+1)$ the subset of such matrices. Each good $j, j \leq n$ is liked by agent j only (or is useless if $w_j = 0$), while good $n+1$ is liked by the first ℓ agents only. The subset $\mathcal{A}(n, n+1) \cap \mathcal{D}(n, n+1)$, denoted $\mathcal{AD}(n, n+1)$, contains the following allocations $z^{\ell,t}$, where $t \in \mathbb{R}_+^{[\ell]}$:

$$z_{ij}^{\ell,t} = \delta_{ij} \text{ for all } i \text{ and } j \leq n \text{ ; } z_{i(n+1)}^{\ell,t} = t_i \text{ if } i \leq \ell, \ = 0 \text{ if } i \geq \ell + 1 \tag{9}$$

(thus $\sum_1^\ell t_i = 1$)

By Efficiency f maps $\mathcal{D}(n, n+1)$ into $\mathcal{AD}(n, n+1)$; if all goods are useful, i. e., $w \gg 0$, then $f(u^{\ell,w})$ reduces to $z^{\ell,t}$.

It is easy by Lemma 2 to compute the allocation $f^{CEEI}(u^{\ell,w}) = z^{\ell,t} \in \mathcal{AD}(n, n+1)$ at such a problem.[8] Relabel the rows so that the sequence w_k is weakly increasing in $[\ell]$ (and arbitrary after ℓ). The inequality

$$w_i \leq \frac{1}{i}(1 + \sum_{j=1}^{i} w_j) \tag{10}$$

holds for $i = 1$ and we let i^* be the largest integer in $[\ell]$ such that it does. Thus if $i^* \leq \ell - 1$ we also have $\frac{1}{i}(1 + \sum_{j=1}^{i} w_j) < w_{i'}$ for $i^* + 1 \leq i' \leq \ell$.

The vector t defining $z^{\ell,t}$, and the utility profile $F^{CEE}(u^{\ell,w}) = U$, are

$$t_i = \frac{1}{i^*}(1 + \sum_{j=1}^{i^*} w_j) - w_i \text{ for } 1 \leq i \leq i^*, \text{ and } t_i = 0 \text{ for } i^* + 1 \leq i \leq \ell$$

$$U_i = \frac{1}{i^*}(1 + \sum_{j=1}^{i^*} w_j) \text{ for } 1 \leq i \leq i^*, \text{ and } U_i = w_i \text{ for } i^* + 1 \leq i \leq n \tag{11}$$

(we omit the easy proof)

The key to the rest of the proof is to show that the rule f coincides with f^{CEEI} on $\mathcal{D}(n, n+1)$. But before checking this, we need a reduction Lemma which is our next step.

Step 3 *Fix $u \in \mathcal{R}(n, m)$, a good $a \in [m]$ useful only to the first ℓ agents, $\ell \geq 1$, and an allocation $z \in f(u)$. Consider the matrix $u^{\ell,w} \in \mathcal{R}(n, n+1)$, where*

$$w_i = \frac{u_i \cdot z_i}{u_{ia}} - z_{ia} \text{ for } 1 \leq i \leq \ell \tag{12}$$

$$w_i = u_i \cdot z_i \text{ for } i \geq \ell + 1 \tag{13}$$

Then we have $f(u^{\ell,w}) = z^{\ell,t}$ (or $z^{\ell,t} \in f(u^{\ell,w})$ if some goods are useless), with $t_i = z_{ia}$ for all $i \leq \ell$.

Proof of the claim. Set $U = F(A, u)$ so $U_i = u_i \cdot z_i$. Starting from problem (A, u) and allocation z, we use EO and Step 1 to construct a sequence of problems $(^kA, ^ku)$ and allocations ^kz such that for all k: $F(^kA, ^ku) = U$ and $^kz \in f(^kA, ^ku)$, and the sequence ends at $([n+1], u^{\ell,w}), z^{\ell,t}$ defined in the Claim.

EO is about merging equivalent goods, but it also allows us to split any good $b \in A$ into equivalent goods. We split b into n equivalent goods b^i, one for each $i \in N$: utilities for b^i are now $(z_{ib}u_{jb})_{j \in N}$; in particular b^i is useless if $z_{ib} = 0$. Write $(^1A, ^1u)$ the new problem with $|A| + n - 1$ goods. By EO $F(^1A, ^1u) = U$ therefore $f(^1A, ^1u)$ contains the allocation ^1z that gives all good b^i to i ($^1z_{ib_j} = \delta_{ij}$) and coincides with z elsewhere. Now let ^2u obtain from ^1u by lowering to zero each term $^1u_{jb^i} = z_{ib}u_{jb}$ with $j \neq i$, while every other entry of ^1u is unchanged: Step 1 (applied to each good b^i) implies $^1z \in f(^2u)$ and $F(^1A, ^2u) = U$.

[8] Strictly speaking $f^{CEEI}(u^{\ell,w})$ is a singleton only if all goods are useful.

After repeating this operation for every good $b \in A \diagdown a$ (recall a is fixed in the statement of the claim), we obtain a problem $({}^3A, {}^3u)$ with $|{}^3A| = 1 + n \cdot (|A| - 1)$, where each good b^i is liked only by agent i (or is useless) and ${}^3u_{ib^i} = z_{ib}u_{ib}$. Moreover $F({}^3A, {}^3u) = U$ and $f({}^3A, {}^3u)$ contains ${}^3z$ allocating a just like z, while each useful good b^i is eaten by i only. Fix now agent i and consider goods b and c in $A \diagdown a$: if ${}^3u_{ib^i}$ and ${}^3u_{ic^i}$ are both positive the goods b^i and c^i are equivalent because only i likes them; if ${}^3u_{ib^i}$ and/or ${}^3u_{ic^i}$ is zero they are still equivalent because a useless good is equivalent to any other good. Thus if we merge b^i and c^i into a good for which i's utility is ${}^3u_{ib^i} + {}^3u_{ic^i} = z_{ib}u_{ib} + z_{ic}u_{ic}$, EO implies that in the new problem the rule still picks the same utility profile U.

Now we merge successively all goods b^i, $b \in A \diagdown a$, into a single good labeled i, for which the utilities are ${}^4u_{ii} = \sum_{A \diagdown a} z_{ib}u_{ib} = U_i - z_{ia}u_{ia}$ and ${}^4u_{ji} = 0$ for $j \neq i$. We do this for all agents and reach a problem $({}^4A, {}^4u)$ where ${}^4A = [n] \cup \{a\}$, ${}^4u_{ij} = \delta^4_{ij}u_{ii}$ for all $i, j \leq n$, and ${}^4u_{ia} = u_{ia}$ for all $i \leq n$. Also, $F({}^4A, {}^4u) = U$ and $f({}^4A, {}^4u)$ contains the allocation ${}^4z = z^{\ell,t}$, where $t_i = z_{ia}$ for all i.

Upon labeling a as $n+1$, we see that ${}^4A \in \mathcal{R}(n, n+1)$. In order to go from ${}^4u$ to $u^{\ell,w}$ (7) with w defined by (12),(13), we divide each row $i, i \leq \ell$, by u_{ia}; we leave alone the rows $j, j \geq \ell + 1$, so their only non zero term is U_i. This rescaling does not affect the allocations selected by f (Definition 1), and the proof of the claim is complete.

Step 4 We show in this step by induction on n that the rule F, f coincides with the CEEI rule on $\mathcal{D}(n, n+1)$.

Step 4.1 $n = 2$. A problem in $\mathcal{D}(2,3)$ with $\ell = 1$ is one where each good is liked by one agent (at most) so all efficient rules coincide. A problem $\mathcal{Q}$ with $\ell = 2$ is

$$u^\circ = \begin{matrix} w_1 & 0 & 1 \\ 0 & w_2 & 1 \end{matrix}$$

For any number $\gamma, 0 < \gamma \leq 1$ consider the problem

$$u = \begin{matrix} w_1 & 0 & \gamma & 1 \\ 0 & w_1 & 1 & \gamma \end{matrix}$$

By EFF and SYM $F(u) = (w_1 + 1) \cdot (1, 1)$, and agent 1 (on the top row) gets the 4th good (4th column) in $f(u)$ (for sure if $\gamma < 1$, and in one $z \in f(u)$ if $\gamma < 1$). By Step 3 this agent gets the 3rd good in

$$u' = \begin{matrix} w_1 & 0 & 1 \\ 0 & \frac{w_1+1}{\gamma} & 1 \end{matrix}$$

For any w_1, w_2 such that $w_1 + 1 \leq w_2$ we can choose γ so that $\frac{w_1+1}{\gamma} = w_2$. We conclude that whenever $|w_1 - w_2| \geq 1$ in problem $\mathcal{Q}$, the "low utility" agent eats all good 3, and $F(u^\circ) = (w_1 + 1, w_2)$. This is also what the CEEI rule does in this case.

Conversely we assume that in $\mathcal{Q}$ agent 1 eats all good 3 and show that $w_1 + 1 \leq w_2$. Fix $\varepsilon > 0$ and split good 3 in two as follows

$$u'' = \begin{matrix} w_1 & 0 & \varepsilon & 1 - \varepsilon \\ 0 & w_2 & \varepsilon & 1 - \varepsilon \end{matrix}$$

By EO the rule still gives goods 3 and 4 to agent 1, so by Step 3 this agent eats the 3rd good in

$$u''' = \begin{matrix} \frac{w_1+1-\varepsilon}{\varepsilon} & 0 & 1 \\ 0 & \frac{w_2}{\varepsilon} & 1 \end{matrix}$$

Now if $w_2 < w_1 + 1$ we have $\frac{w_2}{\varepsilon} + 1 < \frac{w_1+1-\varepsilon}{\varepsilon}$ for small enough ε, therefore agent 2 should get good 3, contradiction.

We pick now $\mathcal{Q}$ such that $|w_1 - w_2| < 1$: we just proved that both agents must eat some of good 3 for instance $z_{13} = \lambda, z_{23} = 1 - \lambda$. We split good 3 in $\mathcal{Q}$ as follows

$$u^{\circ\circ} = \begin{matrix} w_1 & 0 & \lambda & 1-\lambda \\ 0 & w_2 & \lambda & 1-\lambda \end{matrix}$$

and note that $f(u^{\circ\circ})$ contains the allocation where agent 1 eats all good 3 and none of good 4. By step 3 agent 1 still eats all good 3 in

$$u^{\circ\circ} = \begin{matrix} \frac{w_1}{\lambda} & 0 & 1 \\ 0 & \frac{w_2+1-\lambda}{\lambda} & 1 \end{matrix}$$

therefore $\frac{w_1}{\lambda} + 1 \le \frac{w_2+1-\lambda}{\lambda} \Longleftrightarrow w_1 + \lambda \le w_2 + 1 - \lambda$. Exchanging the roles of agents 1 and 2 gives the opposite inequality so we conclude $w_1 + \lambda = w_2 + 1 - \lambda$, i.e., $F(u^\circ) = \frac{1}{2}(w_1 + w_2 + 1) \cdot (1, 1)$ just like the CEEI rule.

Step 4.2 induction argument. We assume F is the CEEI rule in $\mathcal{D}(m, m+1)$ for $m \le n-1$, and pick a problem $u^{\ell,w} \in \mathcal{D}(n, n+1)$ as in (7) (8). If $\ell \le n-1$ we can partition the problem into $N^1 = [\ell], A^1 = [\ell] \cup \{n+1\}$ and $N^2 = \{\ell+1, \cdots, n\}, A^2 = \{\ell+1, \cdots, n\}$: the PAR property and the inductive assumption ensure that F distributes A^1 to N^1 exactly like CEEI, and A^2 to N^2 in the obvious efficient way, so F and F^{CEEI} coincide on $u^{\ell,w}$.

Assume now $\ell = n$ so all agents like good $n+1$. By EFF $f(u^{n,w}) = z^{n,t}$ where t, the allocation t of good $n+1$, is unique. Without loss we label the agents so that t_i is weakly decreasing in i. We let i^* be the largest i such that $t_i > 0$. If $i^* \le n-1$, we invoke PAR: f allocates the goods in $[i^*] \cup \{n+1\}$ exactly like in the smaller problem with those i^*+1 goods and the first i^* agents, so by the inductive assumption, exactly like CEEI, and we are done.

We are left with the case where $t_i > 0$ for all $i \in [n]$. We split now good $n+1$ in two

$$u = \begin{matrix} w_1 & \cdots & 0 & t_1 & 1-t_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & w_n & t_n & 1-t_1 \end{matrix}$$

By EO $F(u^{n,w}) = F(u)$, and $f(u)$ contains an allocation where all good $n+1$ goes to agent 1 while the shares of good $n+2$ are $(0, \frac{t_2}{1-t_1}, \cdots, \frac{t_n}{1-t_1})$. Upon partitioning problem u into $N^1 = \{1\}, A^1 = \{1, n+1\}$ and $N^2 = \{2, \cdots, n\}, A^2 = \{2, \cdots, n\} \cup \{n+2\}$, PAR implies that in the reduced problem (N^2, A^2) the shares of good $n+2$ are $(\frac{t_2}{1-t_1}, \cdots, \frac{t_n}{1-t_1})$ as well. After normalizing utilities we see that at the following problem $\widetilde{u}^{n-1,\widetilde{w}}$

$$\widetilde{u}^{n-1,\widetilde{w}} = \begin{matrix} \frac{w_2}{1-t_1} & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{w_n}{1-t_1} & 1 \end{matrix}$$

our rule f shares the last good as $(\frac{t_2}{1-t_1}, \cdots, \frac{t_n}{1-t_1})$. But by the inductive assumption this is also the choice of CEEI: as all shares are strictly positive, we conclude by (11) that all utilities are equal, i. e., $w_2 + t_2 = \cdots = w_n + t_n$. The choice of agent 1 in the above argument was arbitrary, so by repeating it with another agent, we conclude that $F(u^{n,w})$ gives the same utility $w_i + t_i = \frac{1}{n}(1 + \sum_{j=1}^{n} w_j)$ to all agents. Moreover w_i increases weakly in i, and $w_n \leq \frac{1}{n}(1 + \sum_{j=1}^{n} w_j)$ as required by (10). The proof that $F(u^{n,w}) = F^{CEEI}(u^{n,w})$ is complete.

Step 5 We fix an arbitrary $u \in \mathcal{R}(n,m)$ with associated utility profile $U = F(u)$; we also choose an allocation $z \in f(u)$ and an arbitrary good $a \in [m]$. We need to show that the KKT inequalities (2) corresponding to good a: for all $i \in [n]$ such that $z_{ia} > 0$ we have $\frac{u_{ia}}{U_i} \geq \frac{u_{ja}}{U_j}$ for all j.

If good a is liked by exactly ℓ agents, and exactly i^* of those consume some a at z, relabel the latter as the first i^* in u, followed by the $\ell - i^*$ who like a but do not eat any of it. Let $u^{\ell,w} \in \mathcal{R}(n, n+1)$ be defined as in Step 3 by (12),(13): for $i \in [\ell]$ we have $F_i(u^{\ell,w}) = w_i + z_{ia} = \frac{U_i}{u_{ia}}$. And by step 4 $F(u^{\ell,w}) = F^{CEEI}(u^{\ell,w})$ is given by (11): the first i^* agents end up with the same utility $\frac{1}{i^*}(1 + \sum_{j=1}^{i^*} w_j)$ and every agent in $\{i^* + 1, \cdots, \ell\}$ with a higher utility w_i (it does not matter for this statement that the w_i are not ordered increasingly before or after i^*). Hence $\frac{U_i}{u_{ia}}$ is constant for each i such that $z_{ia} > 0$, and all ratios $\frac{U_i}{u_{ia}}$ for each for $i^* + 1 \leq i \leq \ell$ are larger. And for $i \geq \ell + 1$ we have $\frac{u_{ia}}{U_i} = 0$. This proves (2) for good a as desired.